Chapter 21: Option Valuation

I. The Binomial Option Pricing Model

Intro:

1. Goal: to be able to value options
2. Basic approach: **create portfolio of stock and risk-free bonds with same payoff as option**
3. Law of One Price: **value of the option and portfolio must be the same**
4. How it will help: can use current market prices for stock and risk-free bonds to value options

Note: Analysis is for an option on one share of stock.

=> if want to value an option on X shares, multiply results by X.

A. Two-State Single-Period Model

Note: will start with very simple case of only one period and only two possible stock prices a year from today

1. Reasons for starting with such unrealistic assumptions:

1) easier placer to start than Black-Scholes Option Pricing Model (BSOPM)

=> able to build some intuition about what determines option values
=> possible to see how model is derived without an understanding of stochastic calculus (needed for BSOPM)

2) model works pretty well for very short time horizons

2. Definitions

S = current stock price
S_u = "up" stock price next period
S_d = "down" stock price next period
r_f = risk-free interest rate
K = strike price of option
C_u = value of option if stock goes up
C_d = value of option if stock goes down
Δ = number of shares purchase to create replicating portfolio
B = investment in risk-free bonds to create replicating portfolio

3. Creating a replicating portfolio

Key => want payoff on replicating portfolio at t = 1 to equal payoff on call at t = 1 if the stock price rises or if it falls

$S_u\Delta + (1+r_f)B = C_u$ (21.4a)
$S_d\Delta + (1+r_f)B = C_d$ (21.4b)

=> assume know everything except Δ and B
=> two equations and two unknowns (Δ and B)

$\Delta = \frac{C_u - C_d}{S_u - S_d}$ (21.5a)
$B = \frac{C_d - S_d\Delta}{1 + r_f}$ (21.5b)

=> replicating portfolio: buy Δ shares and invest B in risk-free bonds

Note: see Chapter 21 Supplement for steps

Q: What is value of call?

=> same as replicating portfolio

$C = S\Delta + B$ (21.6)

Ex. Assume a stock currently worth \$19 will be worth either \$26 or \$16 next period. What is the value of a call with a \$15 strike price if the risk free rate is 5%?

Key => create binomial tree with possible payoffs for call and stock

Figure 1

Note: In figure, start with black, solve for blue

t=0

S = 19
K = 15
rf=.05
Δ = 1
B = -14.2857
Cost of portfolio = 4.71
= 19(1) - 14.2857
C = 4.71

t=1

Su = 26
Cu = 11
Portfolio = 1(26) - 14.2857(1.05) = 11

Sd = 16
Cd = 1
Portfolio = 1(16) - 14.2857(1.05) = 1

Video

Using 21.5a: $\Delta = \frac{C_u - C_d}{S_u - S_d} = 1 = \frac{\mathbf{11-1}}{\mathbf{26-16}}$

Using 21.5b: $B = \frac{C_d - S_d\Delta}{1+r_f} = -14.2857 = \frac{\mathbf{1-16(1)}}{\mathbf{1.05}}$

=> to create replicating portfolio, short-sell \$14.2857 of Treasuries today and buy 1 share

Check of payoff on portfolio at t = 1:

If S = 26: C_u = 11 = 26 – 15 = **26(1) + (1.05)(-14.2857)**
If S = 16: C_d = 1 = 16 – 15 = **16(1) + (1.05)(-14.2857)**

Value of call today must equal cost to build portfolio today
=> C = SΔ + B = 4.71 = **19(1) – 14.2857** (equation 21.6)

Note: Worth more than if expires now (or if exercise) = 4 = **max(19-15,0)**

4. An Alternative Approach to the Binomial Model

Keys:

1) stock has a variable payoff
=> use stock to duplicate the difference between the high and low call payoffs

2) bonds have a fixed payoff
=> use bonds to adjust of the total payoff higher or lower (to match option)

Note: Use same example: Assume a stock currently worth $19 will be worth either $26 or $16 next period. What is the value of a call with a $15 strike price if the risk free rate is 5%?

1) Creating differences in portfolio payoffs when stock is high rather than low

a) difference between payoff on call when stock is high rather than low = $10 = **11 – 1**

b) difference between high and low payoff on stock = $10 = **26 – 16**

=> need an entire share of stock to duplicate the difference in payoffs on the call

=> Δ = **1**

2) Matching level of payoffs

Key: At t = 1, need $11 if S = $26 and $1 if S = $16
=> replicating portfolio (which has one share) pays $26 or $16
=> **need to get rid of $15 at t = 1**

Q: What kind of transaction today will required an outflow of $15 next period?

=> **short-sell Treasuries today that mature for $15 next period**
=> **short-sell Treasuries worth $14.2857=** $\frac{\mathbf{15}}{\mathbf{1.05}}$
Q: How does this get rid of $15 next period?

3) Summary:

a) Replicating portfolio: short-sell Treasuries worth \$14.2857 and buy 1 share

b) Payoff on replicating portfolio at t = 1:
If S = \$26: 11 = 26 – 15 = what left from stock after buy to cover Treasuries
If S = \$16: 1 = 16 – 15 = what left from stock after buy to cover Treasuries
c) Cost of portfolio = 19 – 14.2857 = 4.71
d) Same results as when plugged numbers into the equations

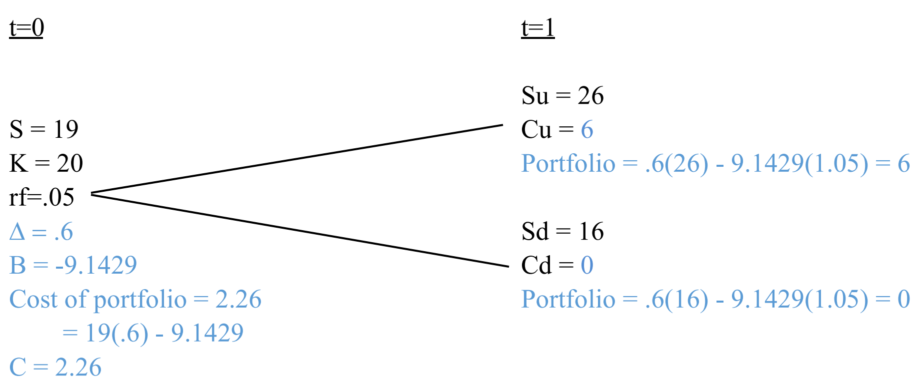
Q: Why does this have to be the price of the call?

Ex. Assume a stock currently worth \$19 will be worth either \$26 or \$16 next period. What is the value of a call with a \$20 strike price if the risk free rate is 5%?

Q: Is the call worth more or less than if the strike price is \$15?

Figure 2

Note: In figure, start with black, solve for blue



Video

1. Using the Equations

Using 21.5a: $\Delta = \frac{C_u - C_d}{S_u - S_d} = .6 = \frac{\mathbf{6-0}}{\mathbf{26-16}}$
Using 21.5b: $B = \frac{C_d - S_d \Delta}{1+r_f} = -9.1429 = \frac{\mathbf{0-16(.6)}}{\mathbf{1.05}}$

=> **short-sell Treasuries worth $9.1429 and buy .6 of a share**

Check of payoff on portfolio at t = 1:

If S = 26: C_u = 6 = 15.6 – 9.6 = **26(.6) + (1.05)(-9.1429)**
If S = 16: = C_d = 0 = 9.6 – 9.6 = **16(.6) + (1.05)(-9.1429)**

Value of call today using 21.6: C = SΔ + B = 2.26 = **19(.6) – 9.1429**

Notes:

1) Value if expires today = 0 **= max (19-20,0)**
2) Value of call if K = 20 ($2.26) is less than if K = 15 ($4.71)

2. Alternative Approach

=> stock will be worth $16 or $26

1) Creating differences in the portfolio payoffs when stock is high rather than low

a) difference between payoff on call when stock is high rather than low = $6 = **6 – 0**

b) difference between high and low payoff on stock = $10 = **26 – 16**

=> portfolio need only $\frac{6}{10}$ of variation in payoff of stock
=> **need $\frac{6}{10}$ of share**

=> Δ = **.6**

Check of difference in payoffs on portfolio at t=1 if Δ = .6:

If S = $26: 15.6 = **.6(26)**
If S = $16: 9.6 = **.6(16)**

=> Difference = 6 = 15.6 – 9.6

2) Matching the level of portfolio payoffs

Key: At t = 1, need $6 (if stock = $26) or $0 (if stock = $16)

=> replicating portfolio (if only include the .6 shares) pays $15.6 or $9.6

=> need to get rid of $9.6

=> **short-sell Treasures today that mature for $9.6 next period**

=> short-sell Treasuries today worth $9.1429 = $\frac{9.6}{1.05}$

Q: How does this get rid of $9.60 next period?

3) Summary:

a) Replicating portfolio: short-sell Treasuries worth \$9.1429 and buy 0.6 shares

b) Payoff on portfolio at t = 1:

If S = \$26: 6 = **.6(26) – 9.6** = what left from stock after buy to cover Treasuries
If S = \$16: 0 = **.6(16) – 9.6** = what left from stock after buy to cover Treasuries

c) Cost of portfolio = 2.26 = 11.4 – 9.1429 = **.6(19) – 9.1429**
=> price of call must also be \$2.26

d) Same results as when plugged numbers into the equations

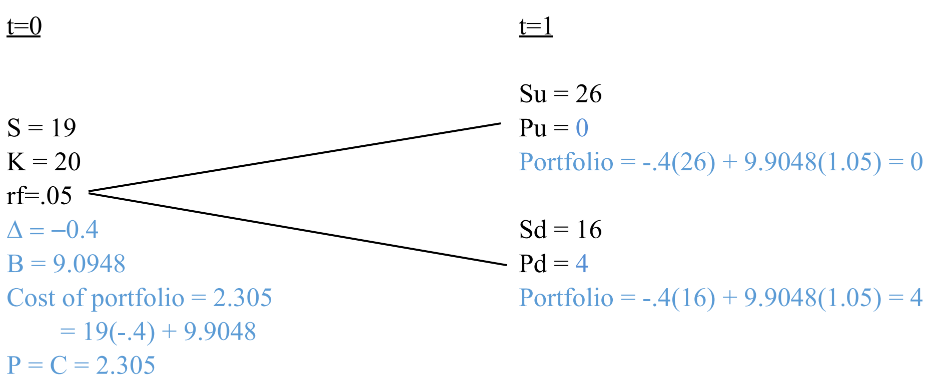
Ex. Assume a stock currently worth \$19 will be worth either \$26 or \$16 next period. What is the value of a put with a \$20 strike price if the risk free rate is 5%?

Key: let C_u and C_d be payoff on put when stock price is up and down (respectively).

=> if you prefer to write them as P_u and P_d feel free to do so.

Figure 3

Note: In figure, start with black, solve for blue



Video

1. Using the Equations

Using 21.5a: $\Delta = \frac{C_u - C_d}{S_u - S_d} = -0.4 = \frac{\mathbf{0-4}}{\mathbf{26-16}}$

Using 21.5b: $B = \frac{C_d - S_d\Delta}{1+r_f} = 9.9048 = \frac{\mathbf{4-16(-0.4)}}{\mathbf{1.05}}$

=> **buy bond for $9.9048 and short-sell 0.4 of a share**

Check of payoff on portfolio at t = 1:

If S = $26: P_u = 0 = – 10.4 + 10.4 = **26(-.4) + (1.05)(9.9048)**
If S = $16: P_d = 4= – 6.4 + 10.4 = **16(-.4) + (1.05)(9.9048)**

Using 21.6: $C = P = S\Delta + B = 2.305 = \mathbf{19(-.4) + 9.9048}$

Note: value if the put expires now = max(20-19,0) = 1

2. Alternative Approach

Note: Stock can end up at $16 or $26

1) Creating differences payoffs when stock is high rather than low

a) difference between payoff on put when stock is high rather than low = – $4 = **0 – 4**

b) difference between high and low payoff on stock = $10 = **26 – 16**

=> when stock is $10 higher, portfolio payoff needs to be $4 lower

Q: What kind of transaction today will lead to a $4 smaller payoff next period if the stock is $10 higher?

=> **short sell 0.4 shares**

Check of difference in payoff on portfolio at t = 1:

If S = $26: - 10.4 = **-.4(26)**
If S = $16: - 6.4= **-.4(16)**

=> difference in payoff = – 4 = **– 10.4 – (– 6.4)**

2) Matching level of payoffs

Key: At t = 1, need \$0 (if stock = \$26) or \$4 (if stock = \$16)

=> replicating portfolio pays – \$10.4 or – \$6.4

=> **always \$10.4 too little**
=> **need to add \$10.4**

=> **buy bond today that matures next year for \$10.4**
=> cost of bond = \$9.9048 = $\frac{10.4}{1.05}$

3) Summary:

a) Replicating portfolio: short-sell 0.4 shares and invest \$9.9048 in Treasuries
b) Payoff on portfolio at t = 1:
If S = \$26: 0 = -.4(26) + 10.4 = what is left from payoff on Treasuries after repurchase stock
If S = \$16: 4 = -.4(16) + 10.4 = what left from payoff on Treasuries after repurchase stock
c) Cost of portfolio = 9.9048 - .4(19) = 9.9048 – 7.6 = 2.305
=> price of put must also be \$2.305
d) Same results as when plugged numbers into the equations

Q: What is the value of the put if K = 15?

=> **zero value since will never be exercised.**

B. A Multiperiod Model

1. Valuing options

=> beginning period, two possible states
=> next period, two possible states from each of these states
=> etc.

Key to solving: **start at end of tree and work back to present**

Ex. Assume that a stock with a current price of \$98 will either increase by 10% or decrease by 5% for each of the next 2 years. If the risk-free rate is 6%, what is the value of a call with a \$100 strike price?

=> possible stock prices at t=1:

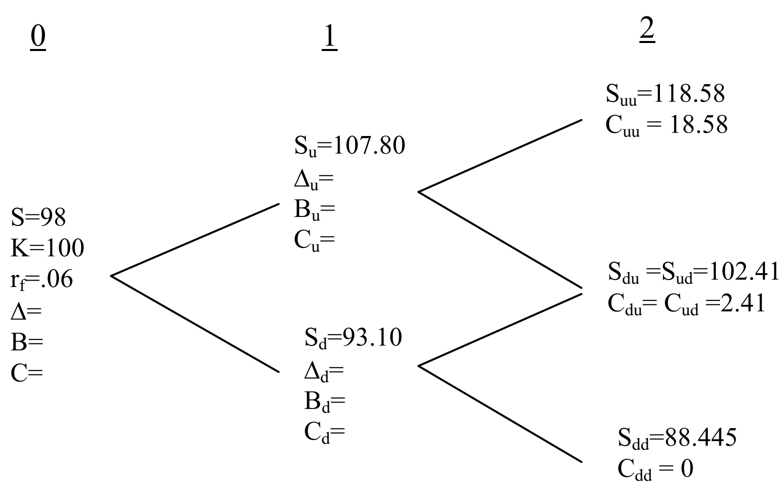
107.80 = **98(1.1)**
93.10 = **98(.95)**

=> possible stock prices at t=2:

118.58 = **98(1.1)²** → $118.58 = 98(1.1)^2$
102.41 = **98(1.1) (.95) =98(.95) (1.1)**
88.445 = $98(.95)^2$

=> possible call values at t=2:

S = 118.58: 18.58 = **max(118.58-100,0)**
S = 102.41: 2.41 = **max(102.41-100,0)**
S = 88.445: 0 = **max(88.445-100,0)**



$$\Delta = \frac{C_u - C_d}{S_u - S_d} \tag{21.5a}$$

$$B = \frac{C_d - S_d\Delta}{1+r_f} \tag{21.5b}$$

$$C = S\Delta + B \tag{21.6}$$

=> Fill in Δ, B, and C on tree for each of the following outcomes

1) t = 1

If S = 107.80:

$$\Delta_u = 1 = \frac{\mathbf{18.58-2.41}}{\mathbf{118.58-102.41}}$$

$$B_u = -94.33962 = \frac{\mathbf{2.41-102.41(1)}}{\mathbf{1.06}}$$

Q: How build replicating portfolio?

C_u = 13.46038 = **107.8(1) – 94.33962**

If S = 93.10:

$$\Delta_d = 0.17257 = \frac{\mathbf{2.41-0}}{\mathbf{102.41-88.445}}$$

$$B_d = -14.39937 = \frac{\mathbf{0-88.445(0.17257)}}{\mathbf{1.06}}$$

Q: How build replicating portfolio?

C_d = 1.66730 = **93.1(.17257) – 14.39937**

2) t = 0 (today):

$$\Delta = 0.80225 = \frac{\mathbf{13.46038-1.6673}}{\mathbf{107.8-93.10}}$$

$$B = -68.8889 = \frac{\mathbf{1.6673-93.1(0.80225)}}{\mathbf{1.06}}$$

C = 9.73167 = **98(.80225) – 68.8889**

Note: To get my numbers, don’t round anything until the final answer.

2. Rebalancing

Key => must rebalance portfolio at t = 1 since Δ and B change at t = 1 when stock price rises or falls

t = 0: S = 98, Δ = 0.80225, B = -68.8889, C = 9.73167

Cost of replicating portfolio = 98(.80225) – 68.8889 = 9.73167

t = 1:

If S = $107.80:

=> value of replicating portfolio = C = 13.46038 = 86.48255 – 73.02234 = **107.8(.80225) – 68.889(1.06)**

=> need Δ = 1

=> change in Δ = .19775 = **1 - .80225**

=> number of shares need to buy/sell: **buy .19775**

=> CF = – 21.3174 = **– .19775 x 107.80**

Q: Where get the cash flow?=> **short-sell Treasuries for $21.3174**

=> B: = - 94.33962 = -73.02223 - 21.3174 = **-68.889(1.06) - 21.3174**

If S = $93.10:

=> value of replicating portfolio = C = 1.66730 = 74.68948 – 73.02234 = **93.10(.80225) – 68.889(1.06)**

=> need Δ = 0.17257

=> change in Δ = - .62968 = **.17257 - .80225**

=> number of shares need to buy/sell: **sell .62968**

=> CF = +58.6232 = **+.62969 x 93.10**

Q: What do with the cash flow?=> **buy to cover bonds worth $58.6232**

=> B: -14.39937 = -73.02223 + 58.6232 = **- 68.8889(1.06) + 58.6232**

3. Payoffs on Replicating Portfolio at t = 2

1) If S = $118.58

Payoff on portfolio = $18.58 = 118.58 – 100 = C_{uu} = **118.58(1) – 94.33962(1.06)**
=> **sell 1 share for $118.58 and buy to cover $100 of bonds.**

2) If S = $102.41

a) If S was $107.80 at t = 1:

Payoff on portfolio = $2.41= C_{ud} = C_{du} = 102.41 – 100 = **102.41(1) – 94.33962(1.06)**
=> **sell share for 102.41 and buy to cover $100 of bonds**

b) if S was $93.10 at t = 1:

Payoff on portfolio = $2.41 = 17.6733 – 15.2633 = C_{dd} = **102.41(.17257) – 14.39937(1.06)**
=> **sell 0.17257 shares at $102.41/share and buy to cover $15.2633 of bonds**

3) If S = 88.445

Payoff on portfolio = $0 = C = 15.2633 – 15.2633 = **88.445(.17257) – 14.39937(1.06)**
=> **sell 0.17257 shares at $88.445/share and buy to cover $15.2633 of bonds**

4. Put example

Assume that a stock with a current price of $27 will either increase by $5 or decrease by $4 for each of the next 2 years. If the risk-free rate is 4%, what is the value of a put with a $30 strike price?

a. Valuation of portfolio (and thus put)

=> possible stock prices at t=1:

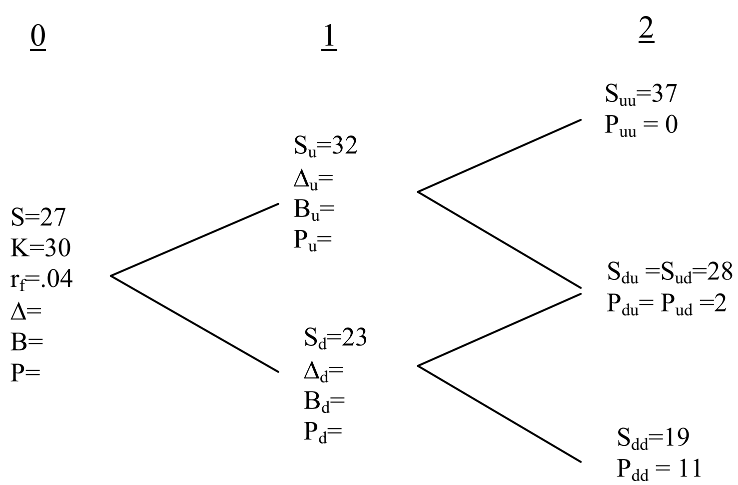
32 = **27 + 5**
23 = **27 – 4**

=> possible stock prices at t=2:

37 = **32 + 5 = 27 + 5 + 5**
28 = **32 – 4 = 23 + 5 = 27 + 5 – 4 = 27 – 4 + 5**
19 = **23 – 4 = 27 – 4 – 4**

=> possible put values at t=2:

S = 37: P = 0 = **max(0, 30 – 37)**
S = 28: P = 2 = **max(0,30 – 28)**
S = 19: P = 11= **max(0,30 – 19)**



$\Delta = \frac{C_u - C_d}{S_u - S_d}$ (21.5a)

$B = \frac{C_d - S_d \Delta}{1 + r_f}$ (21.5b)

C = SΔ + B (21.6)

=> Fill in Δ, B, and C on tree for each of the following outcomes

1) t = 1

If S = 32:

$\Delta_u = -0.22222 = \frac{\mathbf{0-2}}{\mathbf{37-28}}$

$B_u = 7.90598 = \frac{\mathbf{2-28(-0.22222)}}{\mathbf{1.04}}$

Q: How build replicating portfolio?

P_u = 0.79487 = **32(–0.22222) + 7.90598**

If S = 23:

$\Delta_d = -1 = \frac{\mathbf{2-11}}{\mathbf{28-19}}$

$B_d = 28.84615 = \frac{\mathbf{11-19(-1)}}{\mathbf{1.04}}$

Q: How build replicating portfolio?

P_d = 5.84615 = **23(–1) + 28.84615**

2) t = 0 (today):

$\Delta = -0.56125 = \frac{\mathbf{0.79487-5.84615}}{\mathbf{32-23}}$

$B = 18.03364 = \frac{\mathbf{5.84615-23(-0.56125)}}{\mathbf{1.04}}$

P = 2.87979 .= **27(–0.56125) + 18.03364**

Note: To get my numbers, don't round anything until the final answer.

b. Rebalancing of portfolios

Note: To get my numbers, don't round anything

Key => must rebalance portfolio at t = 1

t = 0: S = 27, Δ = –0.56125, B = 18.03364, P = 2.87979

Cost of replicating portfolio = 27(–0.56125) + 18.03364 = 2.87979

t = 1:

If S = 32:

=> value of replicating portfolio = P = 0.79487 = –17.96011 + 18.75499= **32(–0.56125) + 18.03364 (1.04)**

=> need Δ = –0.22222

=> change in Δ = +0.33903 = **–0.22222 – (–0.56125)**

=> number of shares need to buy/sell: **buy to cover .33903 shares**

=> CF= – 10.849 = **– .33903(32)**

Q: Where get the cash flow?=> **sell Treasuries for $10.849**

=> B: 7.90598 = 18.75499 – 10.849 = **18.03364(1.04) – 10.84902**

If S = 23:

=> value of replicating portfolio = P = 5.84615 = –12.90883 +18.75499 = **23(–0.56125) + 18.03364 (1.04)**

=> need Δ = –1

=> change in Δ = – 0.43875 = **–1 – (–0.56125)**

=> number of shares need to buy/sell: **short-sell .43875 shares**

=> CF = +10.09117 = **+.43875(23)**

Q: What do with the cash flow?=> **buy bonds worth $10.09117**

=> B: 28.84615 = **18.03364(1.04) +10.09117**

c. Payoffs on portfolios

1) If S = \$37 at t = 2

Payoff on portfolio = P_{uu} = \$0 = – 8.22222 + 8.22222 = **37(–0.22222) + 7.90598(1.04)**
=> **buy to cover 0.22222 shares with proceeds of bond**

2) If S = \$28 at t = 2

a) If S was \$32 at t = 1:

Payoff on portfolio = P_{ud} = \$2 = – 6.22222 + 8.22222= **28(–0.22222) + 7.90598(1.04)**
=> **receive payoff from bonds and use all but \$2 to buy to cover 0.22222 shares**

b) if S was \$23 at t = 1:

Payoff on portfolio P_{du} = \$2 = – 28 + 30 = **28(–1) + 28.84615(1.04)**
=> **receive payoff from bonds and use all but \$2 to buy to cover 1 share**

3) If S = 19 at t = 2

Payoff on portfolio = P_{dd} = \$11 = – 19 + 30 = **19(–1) + 28.84615(1.04)**
=> **receive payoff from bonds and use all but \$11 to buy to cover 1 share**

II. The Black-Scholes Option Pricing Model

A. European Calls on Non-dividend Paying Stock

$$C = S \times N(d_1) - PV(K) \times N(d_2) \qquad (21.7)$$

where:

$$d_1 = \frac{ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \qquad (21.8a)$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad (21.8b)$$

C = value of call
S = current stock price
N(d) = cumulative normal distribution of d
=> probability that normally distributed variable is less than d
=> Excel function normsdist(d)
PV(K) = present value (price) of a risk-free zero-coupon bond that pays K at the expiration of the option
Note: use risk-free interest rate with maturity closest to expiration of option.

T = years until option expires

σ = annual volatility (standard deviation) of the stock's return over the life of the option

Note: σ is the only variable that must forecast

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The current stock price is $40.75 but is expected to rise to $42 by the time the option expires. The volatility of returns on the firm's stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 2 days = 3.5%, 66 days = 4.8%; 72 days = 5.0%, 79 days = 5.1%. What is the Black-Scholes price for this call?

$\sigma = \mathbf{.21}$

$T = \frac{\mathbf{74}}{\mathbf{365}}$

$\text{PV(K)} = 37.131 = \frac{\mathbf{37.5}}{(\mathbf{1.05})^{74/365}}$

(21.8a) $d_1 = \frac{ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$

$= 1.03089 = \frac{.093004}{.094556} + \frac{.094556}{2} = \frac{\boldsymbol{ln}\left(\frac{\mathbf{40.75}}{\mathbf{37.131}}\right)}{\mathbf{.21}\times\sqrt{\frac{\mathbf{74}}{\mathbf{365}}}} + \frac{\mathbf{.21}\times\sqrt{\frac{\mathbf{74}}{\mathbf{365}}}}{\mathbf{2}}$

(21.8b) $d_2 = d_1 - \sigma\sqrt{T}$

= 0.936337 = **1.03089 -** $\mathbf{.21} \times \sqrt{\frac{\mathbf{74}}{\mathbf{365}}}$

Using Excel: $N(d_1)$ = .848704, $N(d_2)$ = .82545

Notes:

1) calculate N(d) with Excel function "normsdist(d)"
2) feel free to use copy of Excel table to approximate normsdist(d)

Using tables, round d_1 and d_2 to two decimals

$N(d_1)$ = N(1.03) = 0.84849

$N(d_2)$ = N(0.94) = 0.82639

=> close but not exactly the same

(21.7) $C = S \times N(d_1) - PV(K) \times N(d_2)$

= 3.935 = 3.94 = **40.75(.848704) – (37.131)(.82545)**

Note: If use tables, get C = 3.89

B. European Puts on Non-Dividend-Paying Stock

$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)] \tag{21.9}$$

Ex. You are considering purchasing a put that has a strike price of \$37.50 and which expires 74 days from today. The current stock price is \$40.75 but is expected to rise to \$42 by the time the option expires. The volatility of returns on the firm's stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 3 days = 3.5%, 67 days = 4.8%; 73 days = 5.0%, 80 days = 5.1%. What is the Black-Scholes price for this put?

Q: Will the put be more or less valuable than the call? Why?

=> S = 40.75, K = 37.50, PV(K) = 37.131, T = 74/365, σ = .21, r_f = .05, $N(d_1)$ = .848704, $N(d_2)$ = .82545

P = 0.316 = 0.32 = **37.131(1-0.82545) – 40.75(1-0.848704)**

Note: If use tables, P = 0.27

C. Dividend Paying Stocks

Basic idea: subtract from the stock price the present value of dividends between now and expiration of option

=> $S^x = S - PV(Div)$ (21.10)

where:
S = current stock price
PV(Div) = present value of dividends expected prior to expiration of option discounted at the required return on the stock

=> plug S^x, into BSOPM

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The current stock price is $40.75 but is expected to rise to $42 by the time the option expires. The volatility of returns on the firm's stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 3 days = 3.5%, 67 days = 4.8%; 73 days = 5.0%, 80 days = 5.1%. What is the Black-Scholes price for this call if the stock will pay a dividend of $0.25 per share 30 days from today and the required return on the stock is 11% per year?

=> S = 40.75, K = 37.50, PV(K) = 37.131, T = 74/365, σ = .21, $r_f = .05$

$$S^x = 40.502 = \mathbf{40.75} - \frac{\mathbf{.25}}{(\mathbf{1.11})^{\mathbf{30/365}}}$$

Option values

$$d_1 = 0.96637 = \frac{\boldsymbol{ln}\left[\frac{\mathbf{40.502}}{\mathbf{37.131}}\right]}{\mathbf{.21}\sqrt{\frac{\mathbf{74}}{\mathbf{365}}}} + \frac{\mathbf{.21}\sqrt{\frac{\mathbf{74}}{\mathbf{365}}}}{\mathbf{2}}$$; $N(d_1) = 0.83307$; (0.83398 on Table)

$$d_2 = .87181 = \mathbf{0.96637 - .21}\sqrt{\frac{\mathbf{74}}{\mathbf{365}}}$$; $N(d_2) = 0.80834$; (0.80785 on Table)

=> C = **40.502(0.83307) – 37.131(0.80834)** = 3.73 < 3.94 (value if no dividend paid)

=> P = **37.131(1 – 0.80834) – 40.502(1 – 0.83307)** = 0.36 > 0.32 (value if no dividend paid)

Notes:

1) **dividends reduce the value of calls but increase the value of puts**
2) If use tables, C = 3.78 and P = 0.41

D. Standard Form of Black-Scholes

Notes:

1) as far as I know, the following version of BSOPM shows up everywhere except this book
2) source: http://en.wikipedia.org/wiki/Black-Scholes
3) to be consistent with book's symbols, using $N(d_1)$ rather than $\Phi(d_1)$.
4) you are not required to know this version of the model for this class

$$C = S \times N(d_1) - K \times e^{-r \times T} \times N(d_2)$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) \times T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$P = K \times e^{-r \times T} \times [1 - N(d_2)] - S[1 - N(d_1)]$$

Notes:

1) r_f = risk-free rate expressed as effective rate
2) r = risk-free rate expressed as an APR with continuous compounding
3) use the following to convert between APRs and effective rates with continuous compounding:

$$r_f = e^r - 1$$

$$r = \ln(1 + r_f)$$

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The return on a 73-day T-bill (the closest maturity to the call) is 5% per year. The current stock price is $40.75 per share and the stock's volatility is 21%. What is the Black-Scholes price for this call?

Note: same as first Black-Scholes example. Call worth $3.94 and put worth $0.32.

r = ln(1.05) = .04879

$$d_1 = 1.03089 = \frac{ln\left(\frac{40.75}{37.50}\right) + \left(.04879 + \frac{(.21)^2}{2}\right) \times \frac{74}{365}}{.21\sqrt{\frac{74}{365}}}; \text{N}(\text{d}_1) = 0.848704$$

$$d_2 = 0.936337 = 1.03089 - .21\sqrt{\frac{74}{365}}; \text{N}(\text{d}_2) = 0.82545$$

$$C = 40.75 \times 0.848704 - 37.50 \times e^{-.04879 \times \frac{74}{365}} \times 0.82545 = 3.94$$

$$P = 0.32 = 37.50 \times e^{-.04879 \times \frac{75}{365}} \times (1 - .82545) - 40.75 \times (1 - 0.848704)$$

=> same results as with form of model in the book

E. Implied Volatility

Basic idea: can solve for a stock's volatility over the life of the option if know all other variables (including the value of the call)

=> use goal seek in Excel, a Black-Scholes calculator, or trial and error

Ex. What is the implied volatility on a stock given the following information? The price of the call is \$5.75 and the price of the stock on which the call is written is \$45. The call expires 50 days from today and has a strike price of \$40. The return on a 49-day T-bill (the closest maturity to the call) is 4% per year.

Black-Scholes equations:

$$C = S \times N(d_1) - PV(K) \times N(d_2) \qquad (21.7)$$

$$d_1 = \frac{ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \qquad (21.8a)$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad (21.8b)$$

$$PV(K) = 39.786 \ = \frac{40}{(1.04)^{50/365}}$$

$$5.75 = 45 \times N(d_1) - 39.786 \times N(d_2)$$

$$d_1 = \frac{ln\left[\frac{45}{39.786}\right]}{\sigma\sqrt{\frac{50}{365}}} + \frac{\sigma\sqrt{\frac{50}{365}}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{\frac{50}{365}}$$

=> impossible to solve mathematically

Use Excel

=> using goal seek, σ = .3588

F. The Replicating Portfolio

1. Calls

=> can compare Black-Scholes model to binomial model and draw conclusions about how to build a replicating portfolio in a Black-Scholes world

$C = S\Delta + B$ (21.6)
$C = S \times N(d_1) - PV(K) \times N(d_2)$ (21.7)

$\Delta = N(d_1)$ (21.12a)
$B = -PV(K)N(d_2)$ (21.12b)

Ex. What is the replicating portfolio for a call given the following information? The call expires 155 days from today with a strike price of $25. The return on a 154-day T-bill (closest to the expiration of the option) is 2.2%. The stock's current price is $24 and the volatility of the stock over the next 155 days is estimated to be 33%.

$PV(K) = \$24.77 = \frac{\mathbf{25}}{(\mathbf{1.022})^{\mathbf{155/365}}}$

$d_1 = -0.0393 = \frac{\ln\left(\frac{24}{24.77}\right)}{.33\sqrt{\frac{155}{365}}} + \frac{.33\sqrt{\frac{155}{365}}}{2}$; $N(d_1) = .4843$

$\Delta = 0.4843$

$d_2 = -0.2544 = \boldsymbol{d_1 - .33\sqrt{\frac{155}{365}}}$; $N(d_2) = .3996$

B = – 9.90= **– 24.77(0.3996)**

=> **can replicate call on one share of stock by: short-sell Treasuries worth $9.90 and buying .4843 of a share**

Cost of replicating portfolio = cost of option = C = $1.73 = 11.62 – 9.90 = **24(.4843) – 24.77(.3996)**

=> **buying $11.62 of stock for $1.73**
=> **remaining $9.90 comes from short-selling Treasuries**

Note: Replicating portfolio for call will have a long position in the stock and a short position in the bond

=> **a call is equivalent to a levered position in the stock**
=> from Chapter 11 we know that leverage increases risk
=> **a call is riskier than stock itself**

2. Puts

=> comparing (21.6) and (21.9)

$$C = S\Delta + B \quad (21.6)$$
$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)] \quad (21.9)$$

$$\Delta = -[1 - N(d_1)] \quad (21.13a)$$
$$B = PV(K)[1 - N(d_2)] \quad (21.13b)$$

Ex. What is the replicating portfolio for the put in the previous example?

$S = 24$, $K = 25$, $T = 155/365$, $\sigma = .33$, $r_f = .022$, $PV(K) = 24.77$, $N(d_1) = .4843$, $N(d_2) = .3996$, $C = 1.73$, $P = 2.50$

Δ = -0.5157= **– (1 – 0.4843)**
B = 14.8719 = **24.77(1 – 0.3996)**

=> can replicate put on one share by: **short selling .5157 shares worth $12.3768 and buying $14.8719 of risk-free bonds**

=> cost of replicating portfolio = 2.50 = 14.8719 – 12.3768 = **14.8719 - .5157(24)**

Note: **the replicating portfolio for a put will have a short position in the stock and a long position in the bond (lending)**

=> **if stock has positive beta, put's beta will be negative**

III. Risk and Return of an Option

Basic idea: beta of an option equals the beta of its replicating portfolio

Let:
- ΔS = $ invested in stock to create an options replicating portfolio
  - => buy Δ shares at $S per share
- β_S = beta of stock
- B = $ invested in risk-free bonds to create an option's replicating portfolio
- β_B = beta of risk-free bonds

$$\beta_{option} = \beta_{replilcating\ portfolio} = x_S\beta_S + x_B\beta_B = \frac{\Delta S}{\Delta S + B}\beta_S + \frac{B}{\Delta S + B}\beta_B$$

$$\beta_{option} = \frac{\Delta S}{\Delta S + B}\beta_S \text{ since } \beta_B = 0 \quad (21.17)$$

Ex. Assume a call that expires 60 days from today has a strike price equal to the stock's current price of \$15. Assume also that the standard deviation of returns on the stock over the next 60 days is expected to be 30%, and that the risk-free rate over the next 59 days is 4% per year. What is the option's beta if the stock's beta is 1.1? How does the beta change if the stock price rises to \$20 or falls to \$10?

Key: calculate beta of equivalent portfolio of shares of stock and Treasuries

=> equivalent portfolio: buy Δ shares and invest B in bonds

21.12a: $\Delta = N(d_1)$
21.12b: $B = -PV(K)N(d_2)$

$$21.8a: d_1 = \frac{ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

21.8b: $d_2 = d_1 - \sigma\sqrt{T}$

$$\text{PV(K)} = 14.9036 = \frac{\mathbf{15}}{(\mathbf{1.04})^{60/365}}$$

$$d_1 = 0.1138 = \frac{\boldsymbol{ln}\left(\frac{15}{14.9036}\right)}{.3\times\sqrt{\frac{60}{365}}} + \frac{.3\times\sqrt{\frac{60}{365}}}{2}$$

$$d_2 = -0.00781 = \mathbf{0.1138 - .3} \times \sqrt{\frac{\mathbf{60}}{\mathbf{365}}}$$

$N(d_1) = .54531$; $N(d_2) = .496884$

Beta of replicating portfolio:

Investment in Stock = ΔS = 8.179665 = **.54531(15)**
Investment in Treasuries = B = – 7.40536= **-14.9036(.496884)**
Total investment = C = 0.7743 = **8.179665 – 7.40536**

$$\beta_{portfolio} = 11.62 = (10.564)(1.1) + (-9.464)(0) = \left(\frac{\mathbf{8.179665}}{\mathbf{0.7743}}\right)(\mathbf{1.1}) + \left(\frac{\mathbf{-7.40536}}{\mathbf{0.7743}}\right)(\mathbf{0})$$

Use equation 21.17:

=> $\beta_{Call} = \frac{\Delta S}{\Delta S+B}\beta_S = 11.62 = 10.564\ (1.1) = \frac{\mathbf{.545311\times15}}{\mathbf{.545311\times15-7.40536}}(\mathbf{1.1})$

=> if stock price = \$20:

=> $\beta_{Call} = \frac{\Delta S}{\Delta S+B}\beta_S = 4.284 = 3.8944\ (1.1) = \frac{\mathbf{.9934\times20}}{\mathbf{.9934\times20-14.7664}}(\mathbf{1.1})$

Note: call is in the money and less risky

=> if stock price = \$10:

=> $\beta_{Call} = \frac{\Delta S}{\Delta S+B}\beta_S = 34.745 = 31.5864\ (1.1) = \frac{\mathbf{.0006\times10}}{\mathbf{.0006\times10-0.0062}}(\mathbf{1.1})$

Note: call is out of the money and more risky

Note: as an option goes further out of the money, the magnitude (#) of $\frac{\Delta S}{\Delta S+B}$ rises

=> the magnitude of the option's beta rises

Ex. Assume a put has a strike price equal to the stock's current price of \$15. Assume also that standard deviation of returns on the stock over the life of the option is expected to be 30%, that the option expires in 60 days, and that the risk-free rate is 4% per year. What is the option's beta if the stock's beta is 1.1?

Note: Same information as on the call example.

=> $N(d_1)$ = .54531, $N(d_2)$ = .496884, PV(K) = 14.9036

$\beta_{option} = \frac{\Delta S}{\Delta S+B}\beta_S$

Using equations 21.13a and 21.13b for the Δ and B for a put:

21.13a (p. 18): $\Delta = -[1-N(d_1)] = -0.45469 =$ **– [1 – 0.54531]**

21.13b (p. 18): $B = PV(K)[1 - N(d_2)] = 7.49824 =$ **14.9036[1 – 0.496884]**

Beta of replicating portfolio:

Investment in Stock = ΔS = -6.82035 = **-0.45469(15)**
Investment in Treasuries = B = 7.49824 = **14.9036(1 – 0.496884)**
Total investment = P = 0.67789 = **– 6.82035 + 7.49824**

$\beta_{portfolio} = -11.07 = (-10.06)(1.1) + (11.06)0 = \left(\frac{\mathbf{-6.82035}}{\mathbf{0.67789}}\right)(\mathbf{1.1}) + \left(\frac{\mathbf{7.49824}}{\mathbf{0.67789}}\right)(\mathbf{0})$

Using 21.17 (p. 20): $\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = -11.07 = -10.06(1.1) = \frac{-6.82035}{0.67789}(1.1) = \left(\frac{\mathbf{-0.45469(15)}}{\mathbf{-.45469(15)+7.49824}}\right)(\mathbf{1.1})$

Note: if stock price is:

$20 (out of money):

$$\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = -26.84 = -24.404(1.1) = \left(\frac{\mathbf{-0.00659(20)}}{\mathbf{-.00659(20)+0.137153}}\right)(\mathbf{1.1})$$

$10 (in the money):

$$\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = -2.24 = -2.03792(1.1) = \left(\frac{\mathbf{-0.99936(10)}}{\mathbf{-.99936(10)+14.89739}}\right)(\mathbf{1.1})$$

IV. Beta of a Firm's Assets and Risky Debt

Basic idea: Can combine:

1) equation 21.17 (Beta of an option)
2) the idea that an option is equivalent to a portfolio of stocks and risk-free bonds and
3) the idea that stock is essentially a call on the firm's assets

Let:

β_D = beta of firm's risky debt

β_U = beta of firm's unlevered equity = beta of firm's assets

β_E = beta of firm's levered equity

$\Delta = N(d_1)$ when calculate the value of the firm's stock as a call on the firm's assets

A = market value of the firm's assets

D = market value of the firm's debt

E = market value of the firm's equity

$$\beta_D = (1-\Delta)\frac{A}{D}\beta_U = (1-\Delta)\left(1+\frac{E}{D}\right)\beta_U \qquad (21.20)$$

where:

$$\beta_U = \frac{\beta_E}{\Delta\left(1+\frac{D}{E}\right)} \qquad (21.21)$$

Note: derivations of 21.20 and 21.21 in supplement on web

Ex. Assume that the market value of firm's stock is \$100 million and that the beta of the firm's stock is 1.3. Assume also that the firm has issued zero-coupon debt that matures 5 years from today for \$90 million and that the market value of this debt is \$60 million. Assume also that the risk-free rate is 5%. What is the beta of the firm's assets and of the firm's debt?

Notes:

1) Viewing equity as a call on the firm's assets with a strike price of \$90 million (the amount owed the bondholders at maturity in 5 years).
2) When using the Black-Scholes model, we discount the strike price (K) at the risk-free rate
3) To solve for Δ, must:
   a) find σ that causes BSOPM value of stock to equal current market value
   b) determine Δ using this σ

=> A = 160 = **100 + 60**

PV(K) = 70.5174 = $\frac{\mathbf{90}}{(\mathbf{1.05})^5}$

$$d_1 = \frac{ln\left(\frac{160}{70.5174}\right)}{\sigma \times \sqrt{5}} + \frac{\sigma \times \sqrt{5}}{2}$$

$$d_2 = d_1 - \sigma \times \sqrt{5}$$

=> E = 100 = 160 x N(d_1) – 70.5174 x N(d_2)

=> solve for σ that solves for E = 100

Using solver in Excel: σ is .4313, d_1 = 1.33175, N(d_1) = 0.90853, d_2 = 0.36732, N(d_2) = 0.64331

$$\beta_U = \frac{\beta_E}{\Delta\left(1+\frac{D}{E}\right)}; \beta_D = (1 - \Delta)\frac{A}{D}\beta_U$$

$$\beta_U = 0.8943 = \frac{\mathbf{1.3}}{\mathbf{.90853}\left(\mathbf{1}+\frac{\mathbf{60}}{\mathbf{100}}\right)}$$

$$\beta_D = 0.2181 = (\mathbf{1-.90853})\frac{\mathbf{160}}{\mathbf{60}}(\mathbf{.8943})$$

Note: $\beta_A = \beta_U = .8943 = \left(\frac{\mathbf{60}}{\mathbf{160}}\right)(\mathbf{.2181}) + \left(\frac{\mathbf{100}}{\mathbf{160}}\right)(\mathbf{1.3})$