Chapter 21: Option Valuation

I. The Binomial Option Pricing Model

Intro:

1. Goal: to be able to value options
2. Basic approach: create portfolio of stock and risk-free bonds with same payoff as option
3. Law of One Price: value of the option and portfolio must be the same
4. How it will help: can use current market prices for stock and risk-free bonds to value options

Note: Analysis is for an option on one share of stock.

=> if want to value an option on X shares, multiply results by X.

A. Two-State Single-Period Model

Note: will start with very simple case of only one period and only two possible stock prices a year from today

1. Reasons for starting with such unrealistic assumptions:

1) easier placer to start than Black-Scholes Option Pricing Model (BSOPM)

=> able to build some intuition about what determines option values
=> possible to see how model is derived without an understanding of stochastic calculus (needed for BSOPM)

2) model works pretty well for very short time horizons

2. Definitions

S = current stock price
S_u = "up" stock price next period
S_d = "down" stock price next period
r_f = risk-free interest rate
K = strike price of option
C_u = value of option if stock goes up
C_d = value of option if stock goes down
Δ = number of shares purchase to create replicating portfolio
B = investment in risk-free bonds to create replicating portfolio

3. Creating a replicating portfolio

Key => want payoff on replicating portfolio at t = 1 to equal payoff on call at t = 1 if the stock price rises or if it falls

$S_u\Delta + (1+r_f)B = C_u$ (21.4a)
$S_d\Delta + (1+r_f)B = C_d$ (21.4b)

=> assume know everything except Δ and B
=> two equations and two unknowns (Δ and B)

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad (21.5a)$$

$$B = \frac{C_d - S_d\Delta}{1+r_f} \qquad (21.5b)$$

=> replicating portfolio: buy Δ shares and invest B in risk-free bonds

Note: see Chapter 21 Supplement for steps

Q: What is value of call?

=> same as replicating portfolio

$C = S\Delta + B$ (21.6)

Ex. Assume a stock currently worth \$19 will be worth either \$26 or \$16 next period. What is the value of a call with a \$15 strike price if the risk free rate is 5%?

Key => create binomial tree with possible payoffs for call and stock

Figure 1

Note: In figure, start with black, solve for blue

t=0

S = 19
K = 15
rf=.05
Δ = 1
B = -14.2857
Cost of portfolio = 4.71
= 19(1) - 14.2857
C = 4.71

t=1

Su = 26
Cu = 11
Portfolio = 1(26) - 14.2857(1.05) = 11

Sd = 16
Cd = 1
Portfolio = 1(16) - 14.2857(1.05) = 1

Video

Using 21.5a: $\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{11-1}{26-16} = 1$

Using 21.5b: $B = \frac{C_d - S_d \Delta}{1+r_f} = \frac{1-16(1)}{1.05} = -14.2857$

=> to create replicating portfolio, short-sell \$14.2857 of Treasuries today and buy 1 share

Check of payoff on portfolio at t = 1:

If S = 26: 26(1) + (1.05)(-14.2857) = 26 – 15 = 11 = C_u
If S = 16: 16(1) + (1.05)(-14.2857) = 16 – 15 = 1 = C_d

Value of call today must equal cost to build portfolio today
=> C = SΔ + B = 19(1) – 14.2857 = 4.71 (equation 21.6)

Note: Worth more than if expires now (or if exercise) = max(19-15,0) = 4

4. An Alternative Approach to the Binomial Model

Keys:

1) stock has a variable payoff
   => use stock to duplicate the difference between the high and low call payoffs

2) bonds have a fixed payoff
   => use bonds to adjust of the total payoff higher or lower (to match option)

Note: Use same example: Assume a stock currently worth $19 will be worth either $26 or $16 next period. What is the value of a call with a $15 strike price if the risk free rate is 5%?

1) Creating differences in portfolio payoffs when stock is high rather than low

a) difference between payoff on call when stock is high rather than low = $10 = 11 – 1

b) difference between high and low payoff on stock = $10 = 26 – 16

=> need an entire share of stock to duplicate the difference in payoffs on the call

=> $\Delta = 1$

2) Matching level of payoffs

Key: At t = 1, need $11 if S = $26 and $1 if S = $16
=> replicating portfolio (which has one share) pays $26 or $16
=> need to get rid of $15 at t = 1

Q: What kind of transaction today will required an outflow of $15 next period?

=> short-sell Treasuries today that mature for $15 next period

=> short-sell Treasuries worth $\frac{15}{1.05}$ = $14.2857

Q: How does this get rid of $15 next period?

3) Summary:

a) Replicating portfolio: short-sell Treasuries worth $14.2857 and buy 1 share

b) Payoff on replicating portfolio at t = 1:

If S = $26: 11 = 26 – 15 = what left from stock after buy to cover Treasuries

If S = $16: 1 = 16 – 15 = what left from stock after buy to cover Treasuries

c) Cost of portfolio = 19 – 14.2857 = 4.71

d) Same results as when plugged numbers into the equations

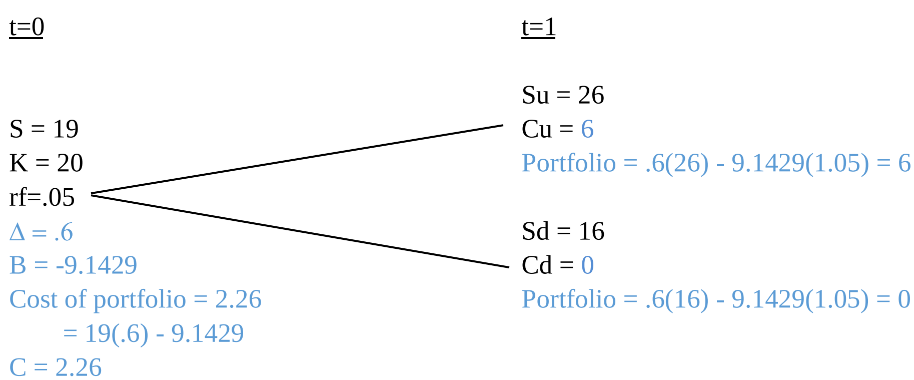
Q: Why does this have to be the price of the call?

Ex. Assume a stock currently worth $19 will be worth either $26 or $16 next period. What is the value of a call with a <u>$20</u> strike price if the risk free rate is 5%?

Q: Is the call worth more or less than if the strike price is $15?

<u>Figure 2</u>

Note: In figure, start with black, solve for blue



[Video]

1. Using the Equations

Using 21.5a: $\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{6-0}{26-16} = .6$

Using 21.5b: $B = \frac{C_d - S_d\Delta}{1+r_f} = \frac{0-16(.6)}{1.05} = -9.1429$

=> short-sell Treasuries worth $9.1429 and buy .6 of a share

Check of payoff on portfolio at t = 1:

If S = 26: 26(.6) + (1.05)(-9.1429) = 15.6 – 9.6 = 6 = C_u
If S = 16: 16(.6) + (1.05)(-9.1429) = 9.6 – 9.6 = 0 = C_d

Value of call today using 21.6: C = SΔ + B = 19(.6) – 9.1429 = 2.26

Notes:

1) Value if expires today = max (19-20,0) = 0
2) Value of call if K = 20 ($2.26) is less than if K = 15 ($4.71)

2. Alternative Approach

=> stock will be worth $16 or $26

1) Creating differences in the portfolio payoffs when stock is high rather than low

a) difference between payoff on call when stock is high rather than low = $6 = 6 – 0

b) difference between high and low payoff on stock = $10 = 26 – 16

=> portfolio need only $\frac{6}{10}$ of variation in payoff of stock

=> need $\frac{6}{10}$ of share

=> Δ = .6

Check of difference in payoffs on portfolio at t=1 if Δ = .6:

If S = $26: .6(26) = 15.6
If S = $16: .6(16) = 9.6

=> Difference = 15.6 – 9.6 = 6

2) Matching the level of portfolio payoffs

Key: At t = 1, need $6 (if stock = $26) or $0 (if stock = $16)

=> replicating portfolio (if only include the .6 shares) pays $15.6 or $9.6

=> need to get rid of $9.6

=> short-sell Treasures today that mature for $9.6 next period

=> short-sell Treasuries today worth $\frac{9.6}{1.05}$ = $9.1429

Q: How does this get rid of $9.60 next period?

3) Summary:

a) Replicating portfolio: short-sell Treasuries worth $9.1429 and buy 0.6 shares

b) Payoff on portfolio at t = 1:

If S = $26: 6 = .6(26) – 9.6 = what left from stock after buy to cover Treasuries

If S = $16: 0 = .6(16) – 9.6 = what left from stock after buy to cover Treasuries

c) Cost of portfolio = .6(19) – 9.1429 = 11.4 – 9.1429 = 2.26
=> price of call must also be $2.26

d) Same results as when plugged numbers into the equations

Ex. Assume a stock currently worth \$19 will be worth either \$26 or \$16 next period. What is the value of a put with a \$20 strike price if the risk free rate is 5%?

Key: let C_u and C_d be payoff on put when stock price is up and down (respectively).

=> if you prefer to write them as P_u and P_d feel free to do so.

Figure 3

Note: In figure, start with black, solve for blue

t=0	t=1
	Su = 26
S = 19	Pu = 0
K = 20	Portfolio = -.4(26) + 9.9048(1.05) = 0
rf=.05	
Δ = –0.4	Sd = 16
B = 9.0948	Pd = 4
Cost of portfolio = 2.305	Portfolio = -.4(16) + 9.9048(1.05) = 4
= 19(-.4) + 9.9048	
P = C = 2.305	

Video

1. Using the Equations

Using 21.5a: $\Delta = \dfrac{C_u - C_d}{S_u - S_d} = \dfrac{0\text{-}4}{26-16} = \text{-}0.4$

Using 21.5b: $B = \dfrac{C_d - S_d\Delta}{1+r_f} = \dfrac{4-16(-0.4)}{1.05} = 9.9048$

=> buy bond for \$9.9048 and short-sell 0.4 of a share

Check of payoff on portfolio at t = 1:

If S = \$26: 26(-.4) + (1.05)(9.9048) = – 10.4 + 10.4 = 0 = C_u
If S = \$16: 16(-.4) + (1.05)(9.9048) = – 6.4 + 10.4 = 4 = C_d

Using 21.6: $C \ = \ P = \ S\Delta + B = 19(-.4) + 9.9048 = 2.305$

Note: value if the put expires now = max(20-19,0) = 1

2. Alternative Approach

Note: Stock can end up at \$16 or \$26

1) Creating differences payoffs when stock is high rather than low

a) difference between payoff on put when stock is high rather than low = – \$4 = 0 – 4

b) difference between high and low payoff on stock = \$10 = 26 – 16

=> when stock is \$10 higher, portfolio payoff needs to be \$4 lower

Q: What kind of transaction today will lead to a \$4 smaller payoff next period if the stock is \$10 higher?

=> short sell 0.4 shares

Check of difference in payoff on portfolio at t = 1:

If S = \$26: -.4(26) = -10.4
If S = \$16: -.4(16) = -6.4

=> difference in payoff = – 10.4 – (– 6.4) = – 4

2) Matching level of payoffs

Key: At t = 1, need \$0 (if stock = \$26) or \$4 (if stock = \$16)

=> replicating portfolio pays – \$10.4 or – \$6.4

=> always \$10.4 too little
=> need to add \$10.4

=> buy bond today that matures next year for \$10.4

=> cost of bond = $\frac{10.4}{1.05}$ = \$9.9048

3) Summary:

a) Replicating portfolio: short-sell 0.4 shares and invest $9.9048 in Treasuries

b) Payoff on portfolio at t = 1:

If S = $26: 0 = -.4(26) + 10.4 = what is left from payoff on Treasuries after repurchase stock

If S = $16: 4 = -.4(16) + 10.4 = what left from payoff on Treasuries after repurchase stock

c) Cost of portfolio = 9.9048 - .4(19) = 9.9048 – 7.6 = 2.305

=> price of put must also be $2.305

d) Same results as when plugged numbers into the equations

Q: What is the value of the put if K = 15?

=> zero value since will never be exercised.

B. A Multiperiod Model

1. Valuing options

=> beginning period, two possible states
=> next period, two possible states from each of these states
=> etc.

Key to solving: start at end of tree and work back to present

Ex. Assume that a stock with a current price of $98 will either increase by 10% or decrease by 5% for each of the next 2 years. If the risk-free rate is 6%, what is the value of a call with a $100 strike price?

=> possible stock prices at t=1:

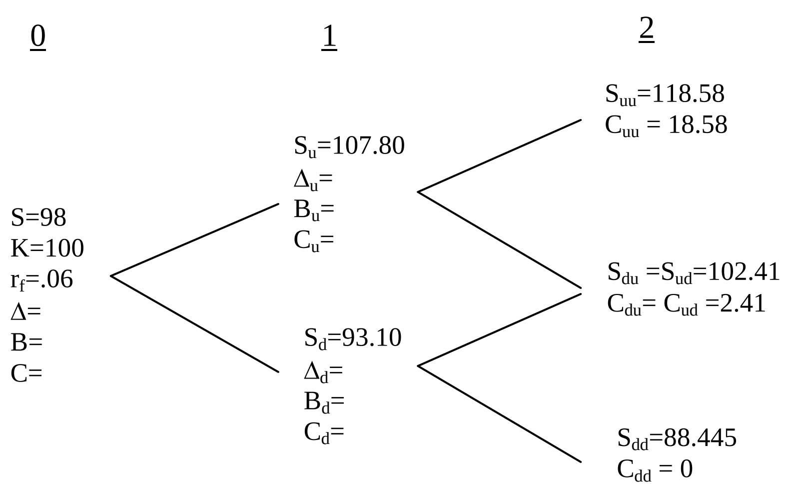
107.80 = 98(1.1)
93.10 = 98(.95)

=> possible stock prices at t=2:

$118.58 = 98(1.1)^2$
102.41 = 98(1.1) (.95) =98(.95) (1.1)
$88.445 = 98(.95)^2$

=> possible call values at t=2:

S = 118.58: 18.58 = max(118.58-100,0)
S = 102.41: 2.41 = max(102.41-100,0)
S = 88.445: 0 = max(88.445-100,0)



$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad (21.5a)$$

$$B = \frac{C_d - S_d \Delta}{1 + r_f} \qquad (21.5b)$$

$$C = S\Delta + B \qquad (21.6)$$

=> Fill in Δ, B, and C on tree for each of the following outcomes

1) t = 1

If S = 107.80:

$$\Delta_u = \frac{18.58 - 2.41}{118.58 - 102.41} = 1$$
$$B_u = \frac{2.41 - 102.41(1)}{1.06} = -94.33962$$

Q: How build replicating portfolio?

C_u = 107.8(1) – 94.33962 = 13.46038

If S = 93.10:

$$\Delta_d = \frac{2.41\text{-}0}{102.41 - 88.445} = 0.17257$$
$$B_d = \frac{0 - 88.445(0.17257)}{1.06} = \text{-}14.39937$$

Q: How build replicating portfolio?

C_d = 93.1(.17257) – 14.39937 = 1.66730

2) t = 0 (today):

$$\Delta = \frac{13.46038 - 1.6673}{107.8\text{-}93.10} = 0.80225$$
$$B = \frac{1.6673 - 93.1(0.80225)}{1.06} = -68.8889$$
C = 98(.80225) – 68.8889 = 9.73167

Note: To get my numbers, don't round anything until the final answer.

2. Rebalancing

Key => must rebalance portfolio at t = 1 since Δ and B change at t = 1 when stock price rises or falls

t = 0: S = 98, Δ = 0.80225, B = -68.8889, C = 9.73167

Cost of replicating portfolio = 98(.80225) – 68.8889 = 9.73167

t = 1:

If S = $107.80:

=> value of replicating portfolio = 107.8(.80225) – 68.889(1.06) = 86.48255 – 73.02234 = 13.46038 = C

=> need Δ = 1

=> change in Δ = 1 - .80225 = .19775

=> number of shares need to buy/sell: buy .19775

=> CF = – .19775 x 107.80 = – 21.3174

Q: Where get the cash flow?=> short-sell Treasuries for $21.3174

=> B: -68.889(1.06) - 21.3174 = -73.02223 - 21.3174 = - 94.33962

If S = $93.10:

=> value of replicating portfolio = 93.10(.80225) – 68.889(1.06) = 74.68948 – 73.02234 = 1.66730 = C

=> need Δ = 0.17257

=> change in Δ = .17257 - .80225 = - .62968

=> number of shares need to buy/sell: sell .62968

=> CF = +.62969 x 93.10 = +58.6232

Q: What do with the cash flow?=> buy to cover bonds worth $58.6232

=> B: - 68.8889(1.06) + 58.6232 = -73.02223 + 58.6232 = -14.39937

3. Payoffs on Replicating Portfolio at t = 2

1) If S = $118.58

Payoff on portfolio = 118.58(1) – 94.33962(1.06) = 118.58 – 100 = $18.58 = C_{uu}
=> sell 1 share for $118.58 and buy to cover $100 of bonds.

2) If S = $102.41

a) If S was $107.80 at t = 1:

Payoff on portfolio = 102.41(1) – 94.33962(1.06) =102.41 – 100 = $2.41 = C_{ud} = C_{du}
=> sell share for 102.41 and buy to cover $100 of bonds

b) if S was $93.10 at t = 1:

Payoff on portfolio = 102.41(.17257) – 14.39937(1.06) = 17.6733 – 15.2633 = $2.41 = C_{dd}
=> sell 0.17257 shares at $102.41/share and buy to cover $15.2633 of bonds

3) If S = 88.445

Payoff on portfolio = 88.445(.17257) – 14.39937(1.06) = 15.2633 – 15.2633 = $0 = C
=> sell 0.17257 shares at $88.445/share and buy to cover $15.2633 of bonds

4. Put example

Assume that a stock with a current price of \$27 will either increase by \$5 or decrease by \$4 for each of the next 2 years. If the risk-free rate is 4%, what is the value of a put with a \$30 strike price?

a. Valuation of portfolio (and thus put)

=> possible stock prices at t=1:

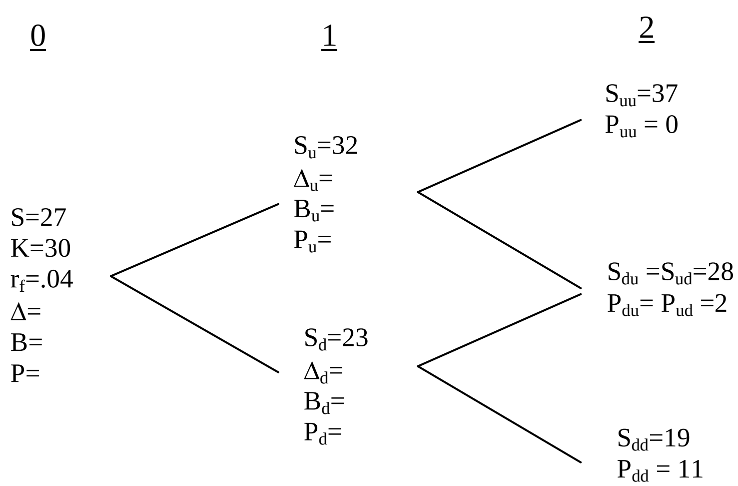
32 = 27 + 5
23 = 27 – 4

=> possible stock prices at t=2:

37 = 32 + 5 = 27 + 5 + 5
28 = 32 – 4 = 23 + 5 = 27 + 5 – 4 = 27 – 4 + 5
19 = 23 – 4 = 27 – 4 – 4

=> possible put values at t=2:

S = 37: P = 0
S = 28: P = 2
S = 19: P = 11



$$\Delta = \frac{C_u - C_d}{S_u - S_d} \qquad (21.5a)$$

$$B = \frac{C_d - S_d \Delta}{1 + r_f} \qquad (21.5b)$$

$$C = S\Delta + B \qquad (21.6)$$

=> Fill in Δ, B, and C on tree for each of the following outcomes

1) t = 1

If S = 32:

$$\Delta_u = \frac{0-2}{37-28} = -0.22222$$
$$B_u = \frac{2-28(-0.22222)}{1.04} = 7.90598$$

Q: How build replicating portfolio?

P_u = 32(–0.22222) + 7.90598 = 0.79487

If S = 23:

$$\Delta_d = \frac{2-11}{28-19} = -1$$
$$B_d = \frac{11-19(-1)}{1.04} = 28.84615$$

Q: How build replicating portfolio?

P_d = 23(–1) + 28.84615 = 5.84615

2) t = 0 (today):

$$\Delta = \frac{0.79487-5.84615}{32-23} = -0.56125$$
$$B = \frac{5.84615-23(-0.56125)}{1.04} = 18.03364$$

P = 27(–0.56125) + 18.03364 = 2.87979

Note: To get my numbers, don’t round anything until the final answer.

b. Rebalancing of portfolios

Note: To get my numbers, don't round anything

Key => must rebalance portfolio at t = 1

t = 0: S = 27, Δ = –0.56125, B = 18.03364, P = 2.87979

Cost of replicating portfolio = 27(–0.56125) + 18.03364 = 2.87979

t = 1:

If S = 32:

=> value of replicating portfolio = 32(–0.56125) + 18.03364 (1.04) = –17.96011 + 18.75499 = 0.79487 = P

=> need Δ = –0.22222

=> change in Δ = –0.22222 – (–0.56125) = +0.33903

=> number of shares need to buy/sell: buy to cover .33903 shares

=> CF = – .33903(32) = – 10.849

Q: Where get the cash flow?=> sell Treasuries for $10.849

=> B: 18.03364(1.04) – 10.84902 = 18.75499 – 10.849 = 7.90598

If S = 23:

=> value of replicating portfolio = 23(–0.56125) + 18.03364 (1.04) = –12.90883 +18.75499 = 5.84615 = P

=> need Δ = –1

=> change in Δ = –1 – (–0.56125) = – 0.43875

=> number of shares need to buy/sell: short-sell .43875 shares

=> CF = +.43875(23) = +10.09117

Q: What do with the cash flow?=> buy bonds worth $10.09117

=> B: 18.03364(1.04) +10.09117 = 28.84615

c. Payoffs on portfolios

1) If S = \$37 at t = 2

Payoff on portfolio = 37(–0.22222) + 7.90598(1.04) = – 8.22222 + 8.22222 = \$0 = P_{uu}
=> buy to cover 0.22222 shares with proceeds of bond

2) If S = \$28 at t = 2

a) If S was \$32 at t = 1:

Payoff on portfolio = 28(–0.22222) + 7.90598(1.04) = – 6.22222 + 8.22222 = \$2 = P_{ud}
=> receive payoff from bonds and use all but \$2 to buy to cover 0.22222 shares

b) if S was \$23 at t = 1:

Payoff on portfolio = 28(–1) + 28.84615(1.04) = – 28 + 30 = \$2 = P_{du}
=> receive payoff from bonds and use all but \$2 to buy to cover 1 share

3) If S = 19 at t = 2

Payoff on portfolio = 19(–1) + 28.84615(1.04) = – 19 + 30 = \$11 = P_{dd}
=> receive payoff from bonds and use all but \$11 to buy to cover 1 share

II. The Black-Scholes Option Pricing Model

A. European Calls on Non-dividend Paying Stock

$$C = S \times N(d_1) - PV(K) \times N(d_2) \tag{21.7}$$

where:

$$d_1 = \frac{\ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \tag{21.8a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{21.8b}$$

C = value of call
S = current stock price
N(d) = cumulative normal distribution of d
=> probability that normally distributed variable is less than d
=> Excel function normsdist(d)

PV(K) = present value (price) of a risk-free zero-coupon bond that pays K at the expiration of the option
Note: use risk-free interest rate with maturity closest to expiration of option.

T = years until option expires

σ = annual volatility (standard deviation) of the stock's return over the life of the option

Note: σ is the only variable that must forecast

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The current stock price is $40.75 but is expected to rise to $42 by the time the option expires. The volatility of returns on the firm's stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 2 days = 3.5%, 66 days = 4.8%; 72 days = 5.0%, 79 days = 5.1%. What is the Black-Scholes price for this call?

$\sigma = .21$

T = $\frac{74}{365}$

PV(K) = $\frac{37.5}{(1.05)^{74/365}} = 37.131$

(21.8a) $d_1 = \frac{\ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$

$$= \frac{\ln\left(\frac{40.75}{37.131}\right)}{.21\times\sqrt{\frac{74}{365}}} + \frac{.21\times\sqrt{\frac{74}{365}}}{2} = \frac{.093004}{.094556} + \frac{.094556}{2} = 1.03089$$

(21.8b) $d_2 = d_1 - \sigma\sqrt{T}$

$= 1.03089 - .21\times\sqrt{\frac{74}{365}} = 0.936337$

Using Excel: $N(d_1) = .848704$, $N(d_2) = .82545$

Notes:

1) calculate N(d) with Excel function “normsdist(d)”
2) feel free to use copy of Excel table to approximate normsdist(d)

Using tables, round d_1 and d_2 to two decimals
$N(d_1) = N(1.03) = 0.84849$
$N(d_2) = N(0.94) = 0.82639$
=> close but not exactly the same

(21.7) $C = S \times N(d_1) - PV(K) \times N(d_2)$
= 40.75(.848704) – (37.131)(.82545) = 3.935 = 3.94

Note: If use tables, get C = 3.89

B. European Puts on <u>Non-Dividend-Paying Stock</u>

$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)] \qquad (21.9)$$

Ex. You are considering purchasing a put that has a strike price of $37.50 and which expires 74 days from today. The current stock price is $40.75 but is expected to rise to $42 by the time the option expires. The volatility of returns on the firm’s stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 3 days = 3.5%, 67 days = 4.8%; 73 days = 5.0%, 80 days = 5.1%. What is the Black-Scholes price for this put?

Q: Will the put be more or less valuable than the call?

=> S = 40.75, K = 37.50, PV(K) = 37.131, T = 74/365, σ = .21, r_f = .05, $N(d_1)$ = .848704, $N(d_2)$ = .82545

P = 37.131(1-0.82545) – 40.75(1-0.848704) = 0.316 = 0.32

Note: If use tables, P = 0.27

C. Dividend Paying Stocks

Basic idea: subtract from the stock price the present value of dividends between now and expiration of option

=> $S^x = S - PV(Div)$ (21.10)

where:
S = current stock price
PV(Div) = present value of dividends expected prior to expiration of option discounted at the required return on the stock

=> plug S^x, into BSOPM

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The current stock price is $40.75 but is expected to rise to $42 by the time the option expires. The volatility of returns on the firm's stock over the past year has been 25% but is expected to be 21% over the next 74 days and 19% over the next year. The returns on T-bills vary by maturity as follows: 3 days = 3.5%, 67 days = 4.8%; 73 days = 5.0%, 80 days = 5.1%. What is the Black-Scholes price for this call <u>if the stock will pay a dividend of $0.25 per share 30 days from today and the required return on the stock is 11% per year</u>?

=> S = 40.75, K = 37.50, PV(K) = 37.131, T = 74/365, σ = .21, r_f = .05

$$S^x = 40.75 - \frac{.25}{(1.11)^{30/365}} = 40.502$$

Option values

$$d_1 = \frac{\ln\left[\frac{40.502}{37.131}\right]}{.21\sqrt{\frac{74}{365}}} + \frac{.21\sqrt{\frac{74}{365}}}{2} = 0.96637$$; N(d_1) = 0.83307; (0.83398 on Table)

$$d_2 = 0.96637 - .21\sqrt{\frac{74}{365}} = .87181$$; N(d_2) = 0.80834; (0.80785 on Table)

=> C = 40.502(0.83307) – 37.131(0.80834) = 3.73 < 3.94 (value if no dividend paid)

=> P = 37.131(1 – 0.80834) – 40.502(1 – 0.83307) = 0.36 > 0.32 (value if no dividend paid)

Notes:

1) dividends reduce the value of calls but increase the value of puts
2) If use tables, C = 3.78 and P = 0.41

D. Standard Form of Black-Scholes

Notes:
1) as far as I know, the following version of BSOPM shows up everywhere except this book
2) source: http://en.wikipedia.org/wiki/Black-Scholes
3) to be consistent with book's symbols, using $N(d_1)$ rather than $\Phi(d_1)$.
4) you are not required to know this version of the model for this class

$$C = S \times N(d_1) - K \times e^{-r \times T} \times N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) \times T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$P = K \times e^{-r \times T} \times [1 - N(d_2)] - S[1 - N(d_1)]$$

Notes:

1) r_f = risk-free rate expressed as effective rate
2) r = risk-free rate expressed as an APR with continuous compounding
3) use the following to convert between APRs and effective rates with continuous compounding:

$$r_f = e^r - 1$$

$$r = \ln(1 + r_f)$$

Ex. You are considering purchasing a call that has a strike price of $37.50 and which expires 74 days from today. The return on a 73-day T-bill (the closest maturity to the call) is 5% per year. The current stock price is $40.75 per share and the stock's volatility is 21%. What is the Black-Scholes price for this call?

Note: same as first Black-Scholes example. Call worth $3.94 and put worth $0.32.

r = ln(1.05) = .04879

$$d_1 = \frac{\ln\left(\frac{40.75}{37.50}\right) + \left(.04879 + \frac{(.21)^2}{2}\right) \times \frac{74}{365}}{.21\sqrt{\frac{74}{365}}} = 1.03089$$; $N(d_1) = 0.848704$

$$d_2 = 1.03089 - .21\sqrt{\frac{74}{365}} = 0.936337$$; $N(d_2) = 0.82545$

$$C = 40.75 \times 0.848704 - 37.50 \times e^{-.04879 \times \frac{74}{365}} \times 0.82545 = 3.94$$

$$P = 37.50 \times e^{-.04879 \times \frac{75}{365}} \times (1 - .82545) - 40.75 \times (1 - 0.848704) = 0.32$$

=> same results as with form of model in the book

E. Implied Volatility

Basic idea: can solve for a stock's volatility over the life of the option if know all other variables (including the value of the call)

=> use goal seek in Excel, a TI-83, or trial and error

Ex. What is the implied volatility on a stock given the following information? The price of the call is \$5.75 and the price of the stock on which the call is written is \$45. The call expires 50 days from today and has a strike price of \$40. The return on a 49-day T-bill (the closest maturity to the call) is 4% per year.

Black-Scholes equations:

$$C = S \times N(d_1) - PV(K) \times N(d_2) \tag{21.7}$$

$$d_1 = \frac{\ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \tag{21.8a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{21.8b}$$

$$PV(K) = \frac{40}{(1.04)^{50/365}} = 39.786$$

$$5.75 = 45 \times N(d_1) - 39.786 \times N(d_2)$$

$$d_1 = \frac{\ln\left[\frac{45}{39.786}\right]}{\sigma\sqrt{\frac{50}{365}}} + \frac{\sigma\sqrt{\frac{50}{365}}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{\frac{50}{365}}$$

=> impossible to solve mathematically

Use Excel

=> using goal seek, σ = .3588

F. The Replicating Portfolio

1. Calls

=> can compare Black-Scholes model to binomial model and draw conclusions about how to build a replicating portfolio in a Black-Scholes world

$$C = S\Delta + B \qquad (21.6)$$

$$C = S \times N(d_1) - PV(K) \times N(d_2) \qquad (21.7)$$

$$\Delta = N(d_1) \qquad (21.12a)$$

$$B = -PV(K)N(d_2) \qquad (21.12b)$$

Ex. What is the replicating portfolio for a call given the following information? The call expires 155 days from today with a strike price of $25. The return on a 154-day T-bill (closest to the expiration of the option) is 2.2%. The stock's current price is $24 and the volatility of the stock over the next 155 days is estimated to be 33%.

$$PV(K) = \frac{25}{(1.022)^{155/365}} = \$24.77$$

$$d_1 = \frac{\ln\left(\frac{24}{24.77}\right)}{.33\sqrt{\frac{155}{365}}} + \frac{.33\sqrt{\frac{155}{365}}}{2} = -0.0393; \; N(d_1) = .4843$$

$\Delta = 0.4843$

$$d_2 = d_1 - .33\sqrt{\frac{155}{365}} = -0.2544; \; N(d_2) = .3996$$

$B = -24.77(0.3996) = -9.90$

=> can replicate call on one share of stock by: short-sell Treasuries worth $9.90 and buying .4843 of a share

Cost of replicating portfolio = cost of option = C = 24(.4843) – 9.90 = 11.62 – 9.90 = 24(.4843) – 24.77(.3996) = $1.73

=> buying $11.62 of stock for $1.73

=> remaining $9.90 comes from short-selling Treasuries

Note: Replicating portfolio for call will have a long position in the stock and a short position in the bond

=> a call is equivalent to a levered position in the stock
=> from Chapter 11 we know that leverage increases risk
=> a call is riskier than stock itself

2. Puts

=> comparing (21.6) and (21.9)

$$C = S\Delta + B \quad (21.6)$$
$$P = PV(K)[1 - N(d_2)] - S[1 - N(d_1)] \quad (21.9)$$

$$\Delta = -[1 - N(d_1)] \quad (21.13a)$$
$$B = PV(K)[1 - N(d_2)] \quad (21.13b)$$

Ex. What is the replicating portfolio for the put in the previous example?

$S = 24$, $K = 25$, $T = 155/365$, $\sigma = .33$, $r_f = .022$, $PV(K) = 24.77$, $N(d_1) = .4843$, $N(d_2) = .3996$, $C = 1.73$, $P = 2.50$

$\Delta = -(1 - 0.4843) = -0.5157$
$B = 24.77(1 - 0.3996) = 14.8719$

=> can replicate put on one share by: short selling .5157 shares worth \$12.3768 and buying \$14.8719 of risk-free bonds

=> cost of replicating portfolio = 14.8719 - .5157(24) = 14.8719 – 12.3768 = 2.50

Note: the replicating portfolio for a put will have a short position in the stock and a long position in the bond (lending)

=> if stock has positive beta, put's beta will be negative

III. Risk and Return of an Option

Basic idea: beta of an option equals the beta of its replicating portfolio

Let:

ΔS = $ invested in stock to create an options replicating portfolio

=> buy Δ shares at $S per share

β_S = beta of stock

B = $ invested in risk-free bonds to create an option's replicating portfolio

β_B = beta of risk-free bonds

$$\beta_{option} = \beta_{replilcating\ portfolio} = x_S\beta_S + x_B\beta_B = \frac{\Delta S}{\Delta S + B}\beta_S + \frac{B}{\Delta S + B}\beta_B$$

$$\beta_{option} = \frac{\Delta S}{\Delta S + B}\beta_S \text{ since } \beta_B = 0 \tag{21.17}$$

Ex. Assume a call that expires 60 days from today has a strike price equal to the stock's current price of $15. Assume also that the standard deviation of returns on the stock over the next 60 days is expected to be 30%, and that the risk-free rate over the next 59 days is 4% per year. What is the option's beta if the stock's beta is 1.1? How does the beta change if the stock price rises to $20 or falls to $10?

Key: calculate beta of equivalent portfolio of shares of stock and Treasuries

=> equivalent portfolio: buy Δ shares and invest B in bonds

21.12a: $\Delta = N(d_1)$

21.12b: $B = -PV(K)N(d_2)$

$$21.8a: d_1 = \frac{\ln\left[\frac{S}{PV(K)}\right]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$21.8b: d_2 = d_1 - \sigma\sqrt{T}$$

$$PV(K) = \frac{15}{(1.04)^{60/365}} = 14.9036$$

$$d_1 = \frac{\ln\left(\frac{15}{14.9036}\right)}{.3\times\sqrt{\frac{60}{365}}} + \frac{.3\times\sqrt{\frac{60}{365}}}{2} = 0.1138$$

$$d_2 = 0.1138 - .3\times\sqrt{\frac{60}{365}} = -0.00781$$

$N(d_1) = .54531$; $N(d_2) = .496884$

Beta of replicating portfolio:

Investment in Stock = ΔS = .54531(15) = 8.179665
Investment in Treasuries = B = -14.9036(.496884) = – 7.40536
Total investment = 8.179665 – 7.40536 = 0.7743 = C

$$\beta_{portfolio} = \left(\frac{8.179665}{0.7743}\right)(1.1) + \left(\frac{-7.40536}{0.7743}\right)(0) = (10.564)(1.1) + (-9.464)0 = 11.62$$

Use equation 21.17:

$$\Rightarrow \beta_{Call} = \frac{\Delta S}{\Delta S + B}\beta_S = \frac{.545311 \times 15}{.545311 \times 15 - 7.40536}(1.1) = 10.564\ (1.1) = 11.62$$

=> if stock price = $20:

$$\Rightarrow \beta_{Call} = \frac{\Delta S}{\Delta S + B}\beta_S = \frac{.9934 \times 20}{.9934 \times 20 - 14.7664}(1.1) = 3.8944\ (1.1) = 4.284$$

Note: call is in the money and less risky

=> if stock price = $10:

$$\Rightarrow \beta_{Call} = \frac{\Delta S}{\Delta S + B}\beta_S = \frac{.0006 \times 10}{.0006 \times 10 - 0.0062}(1.1) = 31.5864\ (1.1) = 34.745$$

Note: call is out of the money and more risky

Note: as an option goes further out of the money, the magnitude (#) of $\frac{\Delta S}{\Delta S + B}$ rises

=> the magnitude of the option's beta rises

Ex. Assume a put has a strike price equal to the stock's current price of $15. Assume also that standard deviation of returns on the stock over the life of the option is expected to be 30%, that the option expires in 60 days, and that the risk-free rate is 4% per year. What is the option's beta if the stock's beta is 1.1?

Note: Same information as on the call example.

=> $N(d_1)$ = .54531, $N(d_2)$ = .496884, PV(K) = 14.9036

$$\beta_{option} = \frac{\Delta S}{\Delta S + B}\beta_S$$

Using equations 21.13a and 21.13b for the Δ and B for a put:

21.13a (p. 18): $\Delta = -[1 - N(d_1)] = -[1 - 0.54531] = -0.45469$

21.13b (p. 18): $B = PV(K)[1 - N(d_2)] = 14.9036[1 - 0.496884] = 7.49824$

Beta of replicating portfolio:

Investment in Stock = ΔS = -0.45469(15) = -6.82035

Investment in Treasuries = B = 14.9036(1 – 0.496884) = 7.49824

Total investment = – 6.82035 + 7.49824 = 0.67789 = P

$$\beta_{portfolio} = \left(\frac{-6.82035}{0.67789}\right)(1.1) + \left(\frac{7.49824}{0.67789}\right)(0) = (-10.06)(1.1) + (11.06)0 = -11.07$$

Using 21.17 (p. 20):

$$\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = \left(\frac{-0.45469(15)}{-.45469(15) + 7.49824}\right)(1.1) = \frac{-6.82035}{0.67789}(1.1) = -10.06(1.1) = -11.07$$

Note: if stock price is:

$20 (out of money):

$$\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = \left(\frac{-0.00659(20)}{-.00659(20) + 0.137153}\right)(1.1) = -24.404(1.1) = -26.84$$

$10 (in the money):

$$\beta_{Put} = \frac{\Delta S}{\Delta S + B}\beta_S = \left(\frac{-0.99936(10)}{-.99936(10) + 14.89739}\right)(1.1) = -2.03792(1.1) = -2.24$$

IV. Beta of a Firm's Assets and Risky Debt

Basic idea: Can combine:

1) equation 21.17 (Beta of an option)
2) the idea that an option is equivalent to a portfolio of stocks and risk-free bonds and
3) the idea that stock is essentially a call on the firm's assets

Let:

β_D = beta of firm's risky debt
β_U = beta of firm's unlevered equity = beta of firm's assets
β_E = beta of firm's levered equity
$\Delta = N(d_1)$ when calculate the value of the firm's stock as a call on the firm's assets
A = market value of the firm's assets
D = market value of the firm's debt
E = market value of the firm's equity

$$\beta_D = (1-\Delta)\frac{A}{D}\beta_U = (1-\Delta)\left(1+\frac{E}{D}\right)\beta_U \qquad (21.20)$$

where:

$$\beta_U = \frac{\beta_E}{\Delta\left(1+\frac{D}{E}\right)} \qquad (21.21)$$

Note: derivations of 21.20 and 21.21 in supplement on web

Ex. Assume that the market value of firm's stock is $100 million and that the beta of the firm's stock is 1.3. Assume also that the firm has issued zero-coupon debt that matures 5 years from today for $90 million and that the market value of this debt is $60 million. Assume also that the risk-free rate is 5%. What is the beta of the firm's assets and of the firm's debt?

Notes:

1) Viewing equity as a call on the firm's assets with a strike price of $90 million (the amount owed the bondholders at maturity in 5 years).
2) When using the Black-Scholes model, we discount the strike price (K) at the risk-free rate
3) To solve for Δ, must:
   a) find σ that causes BSOPM value of stock to equal current market value
   b) determine Δ using this σ

=> A = 100 + 60 = 160,

$$PV(K) = \frac{90}{(1.05)^5} = 70.5174$$

$$d_1 = \frac{\ln\left(\frac{160}{70.5174}\right)}{\sigma \times \sqrt{5}} + \frac{\sigma \times \sqrt{5}}{2}$$

$$d_2 = d_1 - \sigma \times \sqrt{5}$$

=> E = 100 = 160 x N(d_1) – 70.5174 x N(d_2)
=> solve for σ that solves for E = 100

Using solver in Excel: σ is .4313, d_1 = 1.33175, N(d_1) = 0.90853, d_2 = 0.36732, N(d_2) = 0.64331

$$\beta_U = \frac{\beta_E}{\Delta\left(1 + \frac{D}{E}\right)}\ ;\ \beta_D = (1 - \Delta)\frac{A}{D}\beta_U$$

$$\beta_U = \frac{1.3}{.90853\left(1 + \frac{60}{100}\right)} = 0.8943$$

$$\beta_D = (1 - .90853)\frac{160}{60}(.8943) = 0.2181$$

Note: $\beta_A = \beta_U = \left(\frac{60}{160}\right)(.2181) + \left(\frac{100}{160}\right)(1.3) = .8943$