Chapter 11 Supplement: Steps and explanations in some of Chapter 11's equations

Note: Don't have to know these…look at if want to follow the math

1. Proof of footnote 2 on p. 370

1.1) Proof that $Cov(A+B,C)=Cov(A,C)+Cov(B,C)$

$$
\begin{aligned}
Cov(A+B,C) &= \frac{1}{T+1}\sum_t \left(\left(A_t + B_t - \overline{A} - \overline{B}\right)\left(C_t - \overline{C}\right)\right) \\
&= \frac{1}{T+1}\sum_t \left(\left(\left(A_t - \overline{A}\right) + \left(B_t - \overline{B}\right)\right)\left(C_t - \overline{C}\right)\right) \\
&= \frac{1}{T+1}\sum_t \left(\left(A_t - \overline{A}\right)\left(C_t - \overline{C}\right) + \left(B_t - \overline{B}\right)\left(C_t - \overline{C}\right)\right) \\
&= \frac{1}{T+1}\sum_t \left(A_t - \overline{A}\right)\left(C_t - \overline{C}\right) + \frac{1}{T+1}\sum_t \left(B_t - \overline{B}\right)\left(C_t - \overline{C}\right) \\
&= Cov(A,C) + Cov(B,C)
\end{aligned}
$$

1.1b) Note: this can be extended to N parts of the 1st term:

$$Cov(A_1 + A_2 + A_3 + A_4 + \ldots + A_N, B) = Cov(A_1, B) + Cov(A_2, B) + Cov(A_3, B) + Cov(A_4, B) + \ldots + Cov(A_N, B)$$

$$\Rightarrow \; Cov\left(\sum_i A_i, B\right) = \sum_i Cov(A_i, B)$$

1.2) Proof that $Cov(mA,B) = mCov(A,B)$

$$Cov(mA,B) = \frac{1}{T+1}\sum_t \left(mA_t - m\overline{A}\right)\left(B_t - \overline{B}\right)$$
$$= \frac{1}{T+1}\sum_t m\left(A_t - \overline{A}\right)\left(B_t - \overline{B}\right)$$
$$= m\frac{1}{T+1}\sum_t \left(A_t - \overline{A}\right)\left(B_t - \overline{B}\right)$$
$$= mCov(A,B)$$

1.2b) Note: Can combine result of 1b) with result from 2):

$$Cov(m_1A_1 + m_2A_2 + m_3A_3 + m_4A_4 + ... + m_NA_N, B) = m_1Cov(A_1,B) + m_2Cov(A_2,B) + ... + m_NCov(A_N,B)$$
$$\Rightarrow Cov\left(\sum_i m_iA_i, B\right) = \sum_i m_iCov(A_i,B)$$

2. Steps in equations 11.7 and 11.8

$$Var(R_P) = Cov(R_p, R_p)$$

$= Cov(x_1R_1 + x_2R_2, R_p)$ — Note: substituting $x_1R_1 + x_2R_2$ for the first R_p term

$= x_1Cov(R_1,R_p) + x_2Cov(R_2,R_p)$ — Note: using result 1.2

$= x_1Cov(R_1, x_1R_1 + x_2R_2) + x_2Cov(R_2, x_1R_1 + x_2R_2)$ — Note: substituting $x_1R_1 + x_2R_2$ for the other R_p term

$= x_1x_1Cov(R_1,R_1) + x_1x_2Cov(R_1,R_2) + x_2x_1Cov(R_2,R_1) + x_2x_2Cov(R_2,R_2)$ — Note: using result 1.2

$= x_1^2Var(R_1) + x_1x_2Cov(R_1,R_2) + x_1x_2Cov(R_1,R_2) + x_2^2Var(R_2)$ — Note: using result from example 11.4 that $Cov(R_i,R_i) = Var(R_i)$

$= x_1^2Var(R_1) + x_2^2Var(R_2) + 2x_1x_2Cov(R_1,R_2)$ — Note: $Cov(R_1,R_2) = Cov(R_2,R_1)$

3. Steps in equation 11.10

$$Var(R_P) = Cov(R_p, R_p)$$
$$= Cov\left(\sum_i x_i R_i, R_p\right)$$ Note: substituting $\sum_i x_i R_i$ for the first R_p term
$$= \sum_i x_i Cov(R_i, R_P)$$ Note: using result 1.2b

4. Steps in equation 11.11

a. $x_i Cov(R_i, R_P) = x_i Cov\left(R_i, \sum_j x_j R_j\right)$ for each i Note: substituting $\sum_j x_j R_j$ for R_p
$$= x_i \sum_j x_j Cov(R_i, R_j)$$ Note: using result 1.2b
$$= \sum_j x_i x_j Cov(R_i, R_j)$$ Note: x_i is a constant with respect to j

b. $Var(R_p) = \sum_i x_i Cov(R_i, R_P)$
$$= \sum_i \sum_j x_i x_j Cov(R_i, R_j)$$ Note: substituting from 4.a

5. Steps in equation 11.12

$$Var(R_p) = \sum_i \sum_j x_i x_j Cov(R_i, R_j)$$ Equation 11.11

$$= \sum_i \sum_j \left(\frac{1}{n}\right)^2 Cov(R_i, R_j)$$ Assume x_i = 1/n for all stocks in portfolio

$$= \left(\frac{1}{n}\right)^2 \sum_i \sum_j Cov(R_i, R_j)$$

$$= \left(\frac{1}{n}\right)^2 \sum_i Var(R_i) + \left(\frac{1}{n}\right)^2 \sum_i \sum_j Cov(R_i, R_j)(\text{for all i} \neq \text{j})$$ Note: separating out variances from covariances, *Cov(R*$_i$*,R*$_i$*) = Var(R*$_i$*)*

$$= \left(\frac{1}{n}\right)^2 \sum_i n\frac{Var(R_i)}{n} + \left(\frac{1}{n}\right)^2 \sum_i \sum_j Cov(R_i, R_j)(\text{for all i} \neq \text{j})$$

$$= \left(\frac{1}{n}\right)^2 n\sum_i \frac{Var(R_i)}{n} + \left(\frac{1}{n}\right)^2 \sum_i \sum_j Cov(R_i, R_j)(\text{for all i} \neq \text{j})$$ Note: There are "n" variance terms

$$= \frac{1}{n}(\text{Average Variance of Individul Stocks}) + \left(\frac{1}{n}\right)^2 \sum_i \sum_j Cov(R_i, R_j)(\text{for all i} \neq \text{j})$$

$$= \frac{1}{n}(\text{Average Variance of Individul Stocks}) + \left(\frac{1}{n}\right)^2 \sum_i \sum_j (n^2 - n)\frac{Cov(R_i, R_j)}{n^2 - n}(\text{for all i} \neq \text{j})$$ Note: There are "n^2 - n" covariance terms

$$= \frac{1}{n}(\text{Average Variance of Individual Stocks}) + \left(\frac{1}{n}\right)^2 (n^2 - n)(\text{Average Covariance between stocks})$$

$$= \frac{1}{n}(\text{Average Variance of Individual Stocks}) + \left(\left(\frac{n}{n}\right)^2 - \left(\frac{n}{n^2}\right)\right)(\text{Average Covariance between stocks})$$

$$= \frac{1}{n}(\text{Average Variance of Individual Stocks}) + \left(1 - \frac{1}{n}\right)(\text{Average Covariance between stocks})$$

6. Steps in solving for $Corr(R_{re},R_P)$ in Example 11.14 on p. 390

$$Corr(R_{re}, R_P) = \frac{Cov(R_{re}, R_P)}{SD(R_{re})SD(R_P)}$$

$$= \frac{Cov(R_{re}, R_O + x_{re}(R_{re} - R_f))}{SD(R_{re})SD(R_P)}$$

$$= \frac{Cov(R_{re}, R_O + x_{re}R_{re} - x_{re}R_f)}{SD(R_{re})SD(R_P)}$$

$$= \frac{Cov(R_{re}, R_O) + Cov(R_{re}, x_{re}R_{re}) + Cov(R_{re}, -x_{re}R_f)}{SD(R_{re})SD(R_P)}$$ Note: using result 1.1b

$$= \frac{Cov(R_{re}, R_O) + x_{re}Cov(R_{re}, R_{re}) - x_{re}Cov(R_{re}, R_f)}{SD(R_{re})SD(R_P)}$$ Note: using 1.2

$$= \frac{x_{re}Var(R_{re}) + Cov(R_{re}, R_O)}{SD(R_{re})SD(R_P)}$$

Notes:

1) $Cov(R_i,R_f) = 0$
2) $Cov(R_i,R_i) = Var(R_i)$

7. Steps in equation 11.20

1) Assume currently own portfolio P and that borrow and invest in asset *i*

Impact on portfolio's excess return = $E(R_i) - r_f$
Impact on portfolio's risk = $SD(R_i) \times Corr(R_i, R_P)$ Note: see Eq. 11.13

Impact on portfolio's Sharpe ratio = $\dfrac{E[R_i] - r_f}{SD(R_i) \times Corr(R_i, R_P)}$

2) Portfolio's Sharpe ratio will improve if:

$$\frac{E[R_i]-r_f}{SD(R_i)\times Corr(R_i,R_P)} > \frac{E[R_P]-r_f}{SD(R_P)}$$

3) Rearranging terms, the portfolio's Sharpe ratio will improve if:

$$E[R_i] > r_f + \frac{SD(R_i)\times Corr(R_i,R_P)}{SD(R_P)} \times \left(E[R_P]-r_f\right)$$

4) Result: Adding *i* to portfolio P will improve portfolio P's Sharpe ratio if:

$$E[R_i] > r_f + \beta_i^P \times \left(E[R_P]-r_f\right)$$

where: $\beta_i^P = \dfrac{SD(R_i)\times Corr(R_i,R_P)}{SD(R_P)}$

Thus: adding i to Portfolio P will improve the portfolio's Sharpe ratio if:

$$E[R_i] > r_i$$

Expected return on asset *i* > required return on asset *i*

where: $r_i = r_f + \beta_i^P \times \left(E[R_P]-r_f\right)$