Chapter 11: Optimal Portfolio Choice and the Capital Asset Pricing Model

Goal: determine the relationship between risk and return

=> Key issue: investors will only invest in riskier assets if expect to earn a higher return on average

Reason: investors tend to be risk averse
Reason: benefit of gain < pain from equal loss

Note: The chapter includes a lot of math and there are several places where the authors skip steps. For all of the places where I thought the skipped steps made following the development difficult, I've added the missing steps. See Chapter 11 supplement for these additional steps.

I. The Expected Return of a Portfolio

Note:

$$x_i = \frac{MVi}{\sum_j MV_j} \tag{11.1}$$

$$R_P = \sum_i x_i R_i \tag{11.2}$$

$$E[R_P] = \sum_i x_i E[R_i] \tag{11.3}$$

where:

x_i = percent of portfolio invested in asset i
MV_i = market value of asset i = number of shares of i owned × price per share of i
$\sum_j MV_j$ = total value of all securities in the portfolio
R_P = realized return on portfolio
R_i = realized return on asset i
$E[R_P]$ = expected return on portfolio
$E[R_i]$ = expected return on asset i

II. The Volatility of a Two-Stock Portfolio

A. Basic idea

1) by combining stocks, reduce risk through diversification

Q: What determines the amount of risk eliminated?
=> if tend to move together, not much risk cancels out.
=> if don't tend to move together, more risk cancels out

2)

=> need to measure amount of common risk in stocks in our portfolio

B. Covariance and Correlation

1. Covariance: $Cov(R_i, R_j) = \frac{1}{T-1}\sum_t (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$ (11.5)

where: T = number of historical returns

Notes:

1)

=>

2)

=>
=> can go in opposite direction as well
= > if covariance positive, tend to move together
=> if covariance negative, tend to move in opposite directions

3) Covariance will be larger if:

-

-

Goal: isolate the relationship part

2. Correlation: $Corr(R_i, R_j) = \frac{Cov(R_i,R_j)}{SD(R_i)SD(R_j)}$ (11.6)

Notes:

1) Same sign as covariance so same interpretation
2)

=>

3) Correlation is always between +1 and -1

=>

=>

Corr = +1: always move exactly together
Corr = -1: always move in exactly opposite directions

4)

C. Portfolio Variance and Volatility

$$Var(R_p) = x_1^2 Var(R_1) + x_2^2 Var(R_2) + 2x_1x_2Cov(R_1, R_2) \quad (11.8)$$

$$Var(R_p) = x_1^2 SD(R_1)^2 + x_2^2 SD(R_2)^2 + 2x_1x_2Corr(R_1, R_2)SD(R_1)SD(R_2) \quad (11.9)$$

Ex. Use the following returns on JPMorganChase (JPM) and General Dynamics (GD) to estimate the covariance and correlation between JPM and GD and the expected return and volatility of returns on a portfolio of $300,000 invested in JPM and $100,000 invested in GD.

	Return on:	
Year	JPM	GD
1	-21%	36%
2	7%	-34%
3	14%	37%
4	-3%	9%
5	23%	18%
6	19%	18%

$Cov(R_{JPM}, R_{GD}) = \frac{1}{T-1}\sum_t (R_{JPM,t} - \bar{R}_{JPM})(R_{GD,t} - \bar{R}_{GD})$

$\bar{R}_{JPM} = 6.50\% = \frac{1}{6}(-21 + 7 + 14 - 3 + 23 + 19)$

$\bar{R}_{GD} = 14.00\% = \frac{1}{6}(36 - 34 + 37 + 9 + 18 + 18)$

$Cov(RJPM, RGD) = -58.6 = \frac{1}{5}\big((-21 - 6.5)(36 - 14) + (7 - 6.5)(-34 - 14) + (14 - 6.5)(37 - 14) + (-3 - 6.5)(9 - 14) + (23 - 6.5)(18 - 14) + (19 - 6.5)(18 - 14)\big)$

$Corr(R_{JPM,}R_{GD}) = \frac{Cov(R_{JPM}, R_{GD})}{SD_{JPM} \times SD_{GD}}$

$Var(R_{JPM}) = 266.30 = \frac{1}{5}\begin{pmatrix}(-21 - 6.5)^2 + (7 - 6.5)^2 + (14 - 6.5)^2 + \\ (-3 - 6.5)^2 + (23 - 6.5)^2 + (19 - 6.5)^2\end{pmatrix}$

$SD(R_{JPM}) = 16.32\% = \sqrt{266.3}$

$Var(R_{GD}) = 674.80 = \frac{1}{5}((36 - 14)^2 + (-34 - 14)^2 + (37 - 14)^2 + (9 - 14)^2 + (18 - 14)^2 + (18 - 14)^2)$

$SD(R_{GD}) = \sqrt{674.8} = 25.98\%$

$Corr(R_{JPM,}R_{GD}) = -0.1382=$

$E(R_p) = 8.375\%=$

$x_{JPM} = .75 =$

$x_{GD} = .25 =$

$$Var(R_p) = x_1^2 Var(R_1) + x_2^2 Var(R_2) + 2x_1x_2 Cov(R_1, R_2) \qquad (11.8)$$
$$Var(R_p) = x_1^2 SD(R_1)^2 + x_2^2 SD(R_2)^2 + 2x_1x_2 Corr(R_1, R_2) SD(R_1) SD(R_2) \qquad (11.9)$$

11.8: $Var(R_P) = 169.99 = (.75)^2 266.30 + (.25)^2 674.80 + 2(.25)(.75)(-58.6)$

11.9: $Var(R_P) = 169.99 = (.75)^2(16.32)^2 + (.25)^2(25.98)^2 + 2(.25)(.75)(-.1382)(16.32)(25.98)$

$SD(R_p) = 13.04\%=$

Notes:

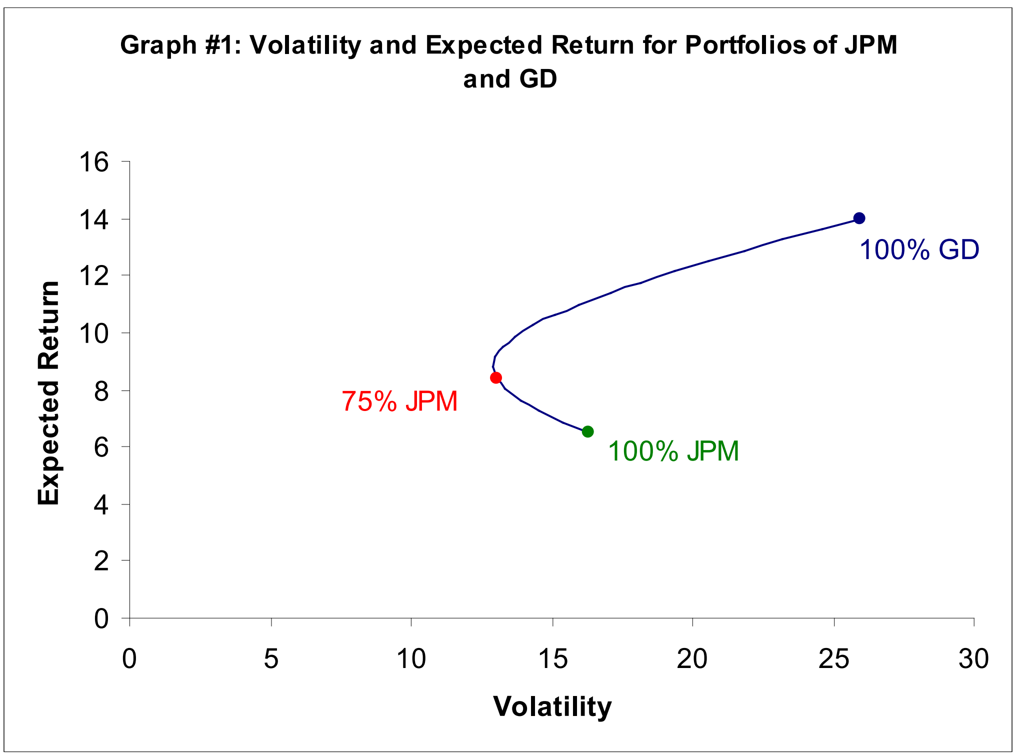
1)

2) can achieve wide range of risk-return combinations by varying portfolio weights

X(JPM)	SD(Rp)	E(Rp)
1.00	16.32	6.50
0.90	14.56	7.25
0.80	13.37	8.00
0.70	12.91	8.75
0.60	13.26	9.50
0.50	14.35	10.25
0.40	16.04	11.00
0.30	18.17	11.75
0.20	20.59	12.50
0.10	23.21	13.25
0.00	25.98	14.00

Q: Why does expected return rise as X_{jpm} falls?
Q: Why does standard deviation initially fall then rise as X_{jpm} falls?

3) the following graph shows the volatility and expected return of various portfolios



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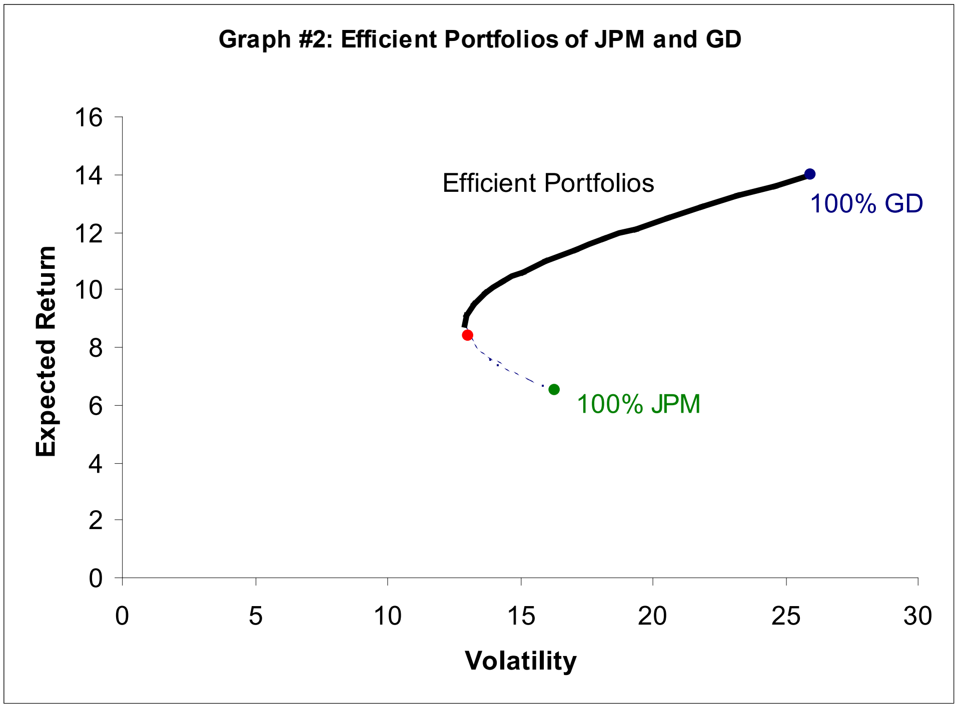
II. Risk Verses Return: Choosing an Efficient Portfolio
Note: Can narrow down our choices a bit

A. Efficient portfolios with two stocks

Efficient portfolio:
Q: Which part of the graph is efficient?

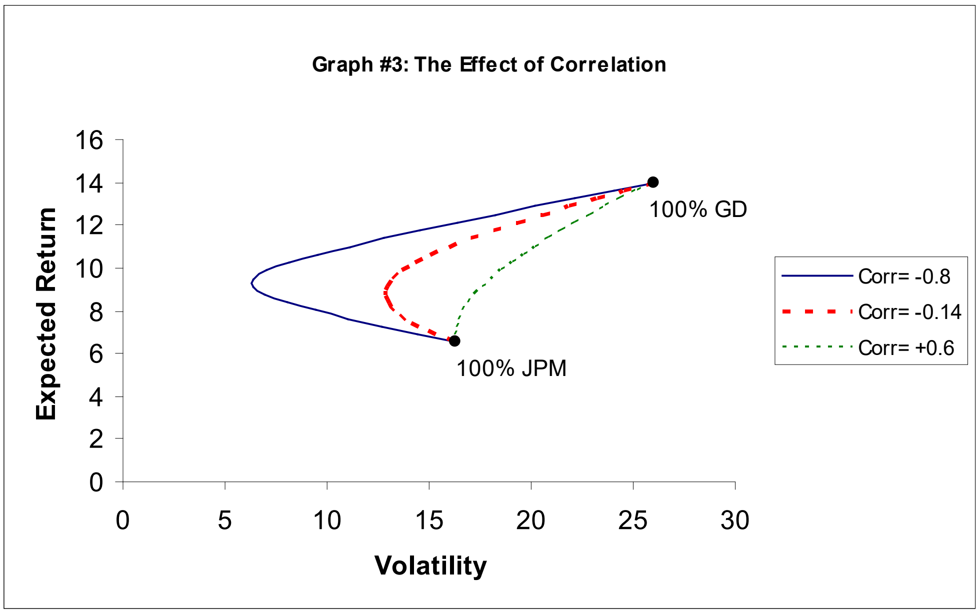


B. The Effect of Correlation

Key: Correlation measures relationship between assets => How impact portfolios?

=>

=>



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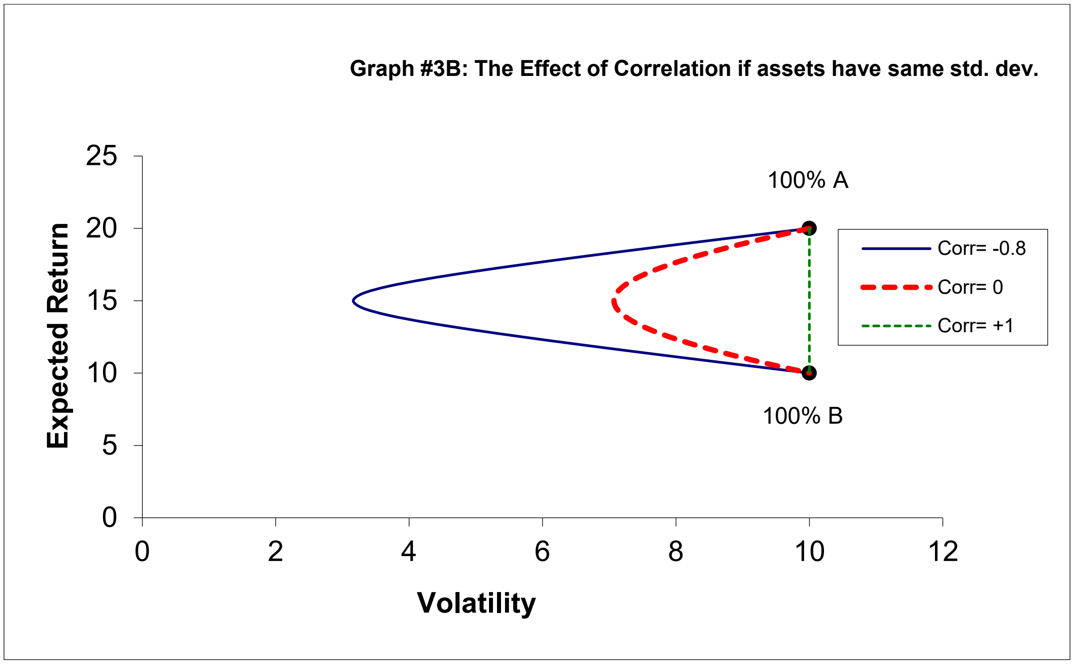
If correlation:

+1: portfolios lie on a straight line between points
-1: portfolios lie on a straight line that "bounces" off vertical axis (risk-free)

Ex. Assume assets A and B have a standard deviation of returns of 10%, but that A has an expected 20% while B has an expected return of 10%. What does graph of possible portfolios look like?



C. Short Sales

1. Short sale: sell stock don't own and buy it back later

Notes:

1) borrow shares from broker (who borrows them from someone who owns the shares)
2) sell shares in open market and receive cash from sale
3) make up any dividends paid on stock while have short position
4) can close out short position at any time by purchasing the shares and returning them to broker
5) broker can ask for shares at any time to close out short position
=> must buy at current market price at that time.
6) until return stock to broker, have short position (negative investment) in stock
7) portfolio weights still add up to 100% even when have short position

Ex. Assume short-sell \$100,000 of JPM and buy \$500,000 of GD. What is volatility and expected return on portfolio if $E(R_{JPM}) = 6.5\%$, $E(R_{GD}) = 14.0\%$; $SD(R_{JPM}) = 16.32\%$, $SD(R_{GD}) = 25.98\%$; and $Corr(R_{JPM}, R_{GD}) = -0.1382$?

Note: total investment = \$400,000 =

x_{GD}= 1.25=

x_{JPM} = -0.25 = -

$E(R_P)$ = 15.875% =

Q: What is allowing us to earn a higher return than 14% (E(R) on GD)?

Notes:

1) Expected dollar gain/loss on JPM= – \$6500 =

2) Expect dollar gain/loss on GD = 70,000 =

= 56,000 + 14,000

=

3) Net expected gain= 63,500 =

4) Expected return = .15875=

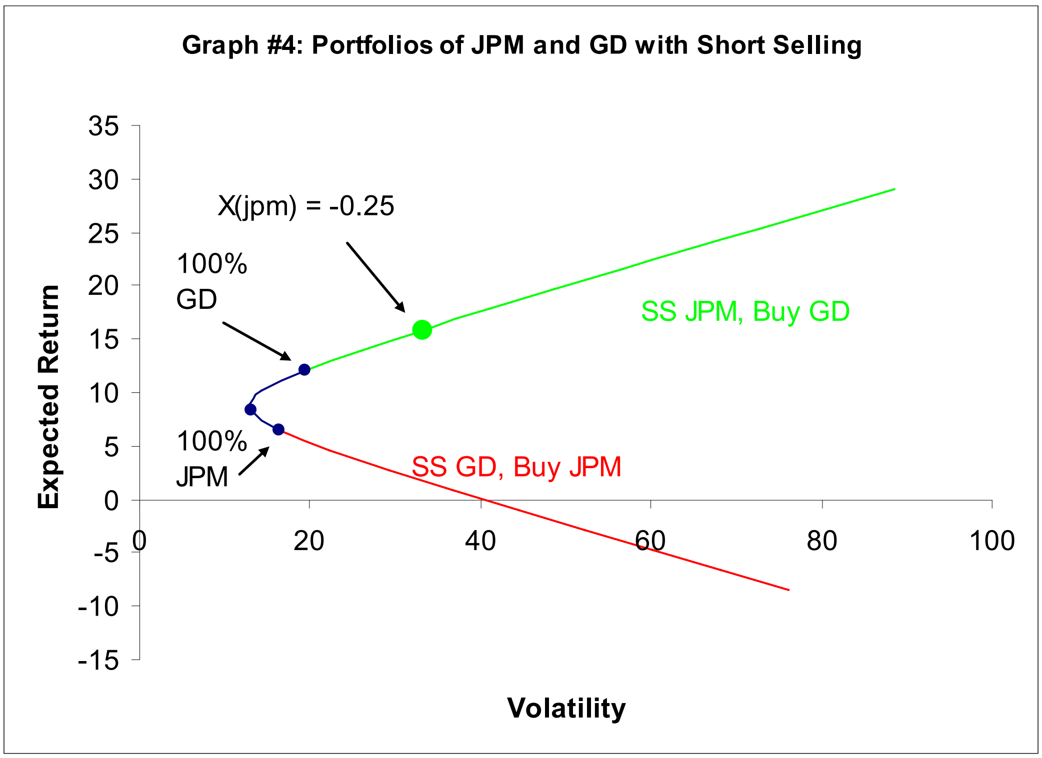
$$Var(R_p) = x_1^2 SD(R_1)^2 + x_2^2 SD(R_2)^2 + 2x_1x_2 Corr(R_1, R_2) SD(R_1) SD(R_2) \quad (11.9)$$

$Var(R_P)$ = 1107.64= $(-.25)^2(16.32)^2 + (1.25)^2(25.98)^2 + 2(-.25)(1.25)(-0.1382)(16.32)(25.98)$

$SD(R_P)$ = 33.28% =

Q: Why is risk higher than simply investing $400,000 in GD (with a standard deviation of returns of 25.98%)?
1) short-selling JPM creates risk
2) gain/loss on a $500,000 investment in GD is greater than the gain/loss on a $400,000 investment in GD
3) loss of diversification:
Correlation between a short and long position in JPM is -1.0
Correlation between short JPM and GD will be +0.1382
=> less diversification than between long position in JPM and GD w/ correlation of -0.1382

2. Impact on graphs => curve extends beyond endpoints (of 100% in one stock or the other).



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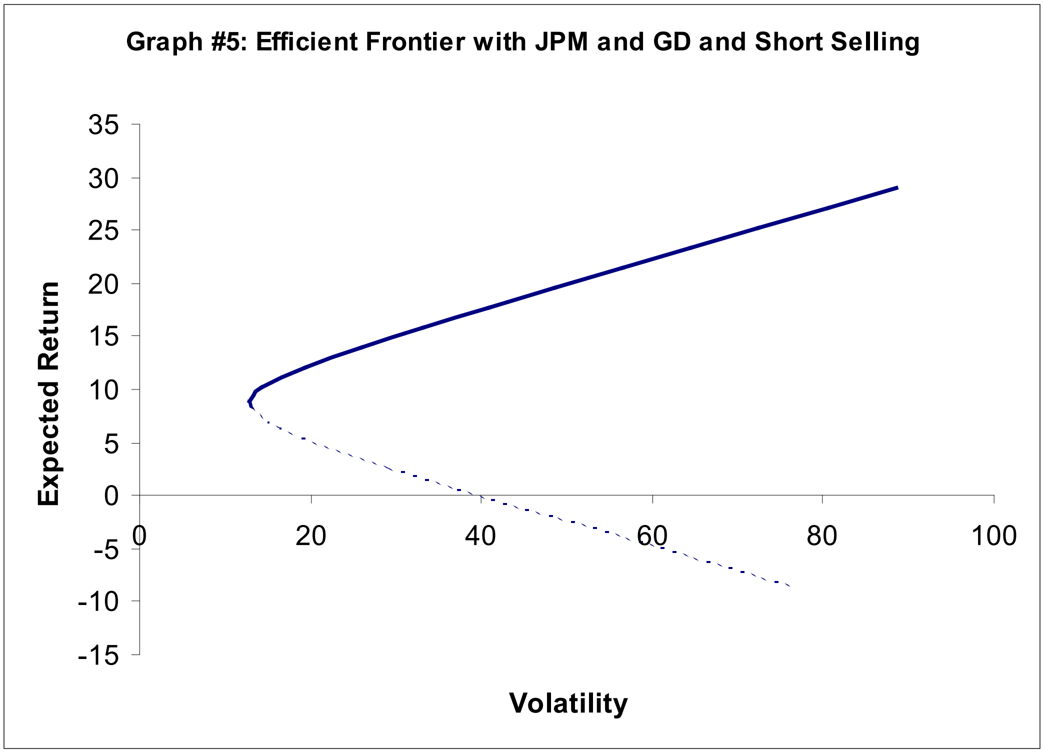
Efficient frontier: portfolios with highest expected return for given volatility
Q: What part of the graph is efficient?



D. Risk Versus Return: Many Stocks

1. Three stock portfolios: long positions only

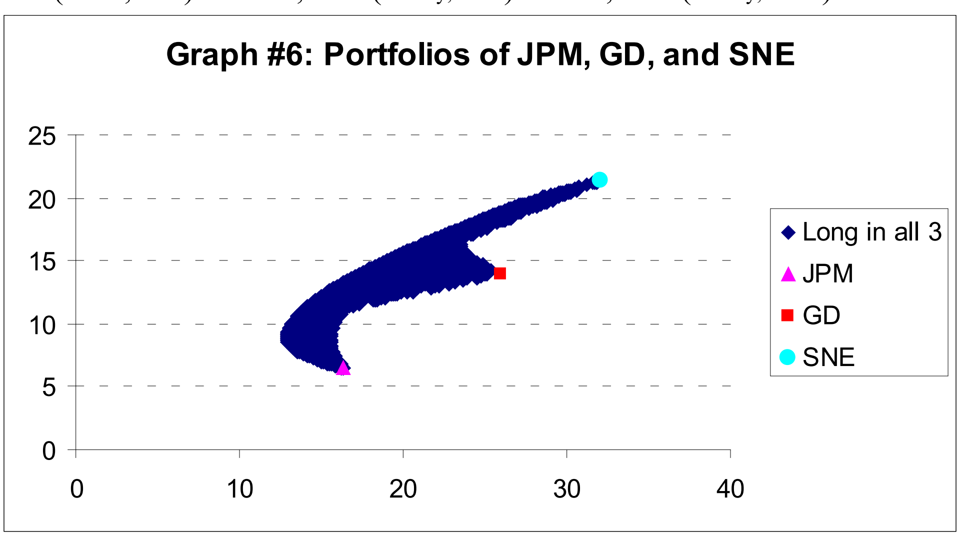
Q: How does adding Sony impact our portfolio?

$E(R_{JPM}) = 6.5\%$, $SD(R_{JPM}) = 16.3\%$;

$E(R_{GD}) = 17\%$, $SD(R_{GD}) = 26\%$;

$E(R_{Sony}) = 21\%$, $SD(R_{Sony}) = 32\%$;

$Corr(R_{JPM},R_{GD}) = -.138$; $Corr(R_{Sony},R_{GD}) = .398$; $Corr(R_{Sony},R_{JPM}) = .204$



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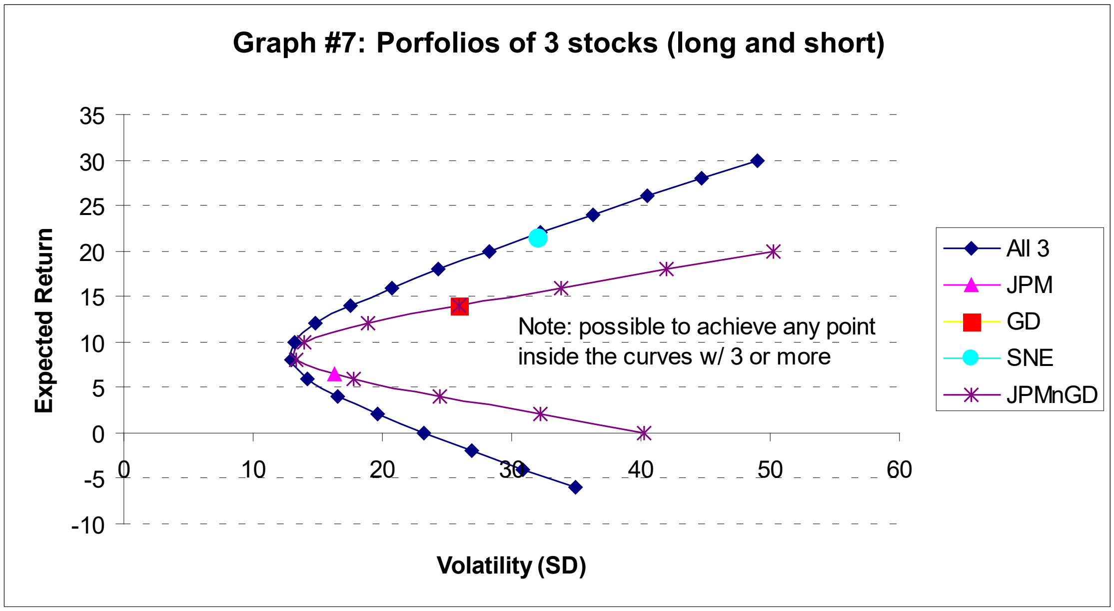
Note: Get area rather than curve when add 3rd asset

Q: How does graph change if allow long and short stock positions?

2. Three Stock Portfolios: long and short positions

Q: What if allow short positions in any of the three stocks?



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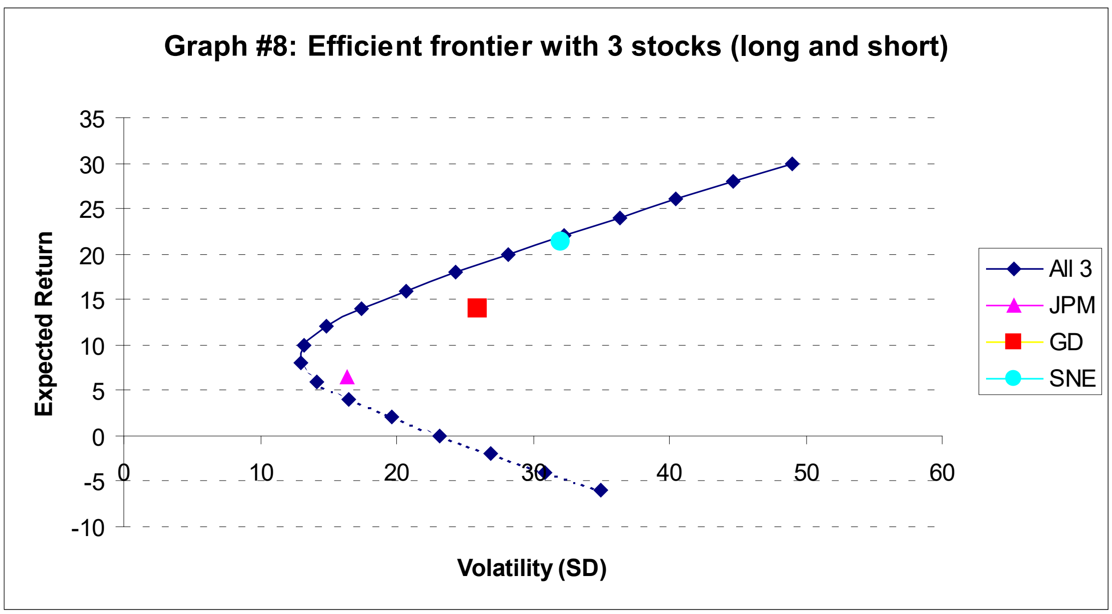


3. More than 3 stocks (long and short):

Note: adding inefficient stock (lower expected return and higher volatility) may improve efficient frontier!

III. Risk-Free Security

A. Ways to change risk

1. Ways to reduce risk

1)

2)

2. Ways to increase risk

1)

2)

Q: Which approach is better?

B. Portfolio Risk and Return

Let:

x = percent of portfolio invested in risky portfolio P
1-x = percent of portfolio invested in risk-free security

1. $E(R_{xP}) = (1-x)r_f + xE(R_P) = r_f + x\big(E(R_P) - r_f\big)$ (11.15)

=> expected return equals risk-free rate plus fraction of risk premium on "P" based on amount we invest in P

2. $SD(R_{xP}) = \sqrt{(1-x)^2 Var\big(r_f\big) + x^2 Var(R_P) + 2(1-x)xCov\big(r_f, R_P\big)}$ (11.16a)

Note: $Var(r_f)$ and $Cov(r_f,R_p)$ both equal 0!

=> $SD(R_{xP}) = xSD(R_P)$ (11.16b)

=> volatility equals fraction of volatility of risky portfolio

3. Note: if increase x, increase risk and return proportionally
=> combinations of risky portfolio P and the risk-free security lie on a straight line between the risk-free security and P.

Ex. Assume that you invest \$80,000 in P (75% JPM and 25% in GD) and \$320,000 in Treasuries earning a 4% return. What volatility and return can you expect? Note: from earlier example: $E(R_P) = 8.375\%$, and $SD(R_P) = 13.04\%$

$x = .2 =$

\$ invested in JPM and GD:

JPM = \$60,000 =

GD = \$20,000=

$SD(R_{.2P}) = 2.61\% =$

$E(R_{.2P}) = 4.88\% =$

Ex. Assume you invest \$360,000 in P and \$40,000 in Treasuries

$x = .9 =$

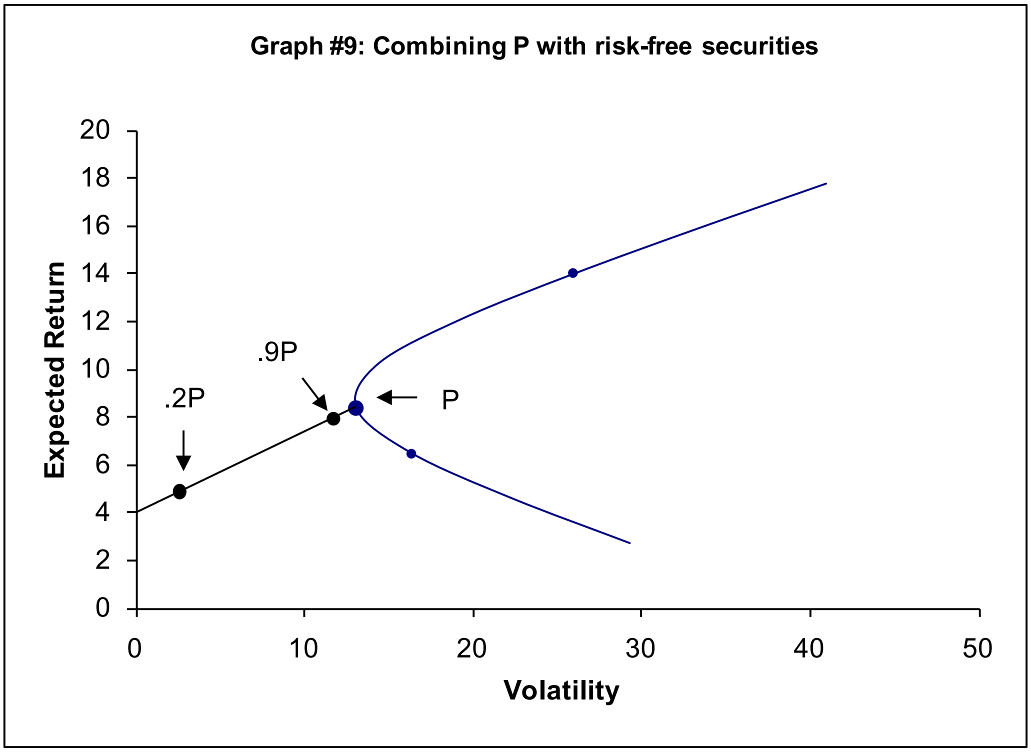
\$ invested in JPM and GD:

JPM = 270,000=

GD 90,000 =

$SD(R_{.9P}) = 11.73\% =$

$E(R_{.9P}) = 7.94\% =$



C. Short-selling the Risk-free Security

Reminder:

x = percent of portfolio invested in risky portfolio P
$1\text{-}x$ = percent of portfolio invested in risk-free security

If $x > 1$ ($x > 100\%$), $1\text{-}x < 0$

=> short-selling risk-free investment

11.16b: $SD(R_{xP}) = xSD(R_P)$
11:15: $E(R_{xP}) = (1-x)r_f + xE(R_P) = r_f + x\big(E(R_P) - r_f\big)$

Ex. Assume that in addition to your \$400,000, you short-sell \$100,000 of Treasuries that earn a risk-free rate of 4% and invest \$500,000 in P. What volatility and return can you expect?

Note: $E(R_P) = 8.375\%$, $SD(R_P) = 13.04\%$

x = 1.25=

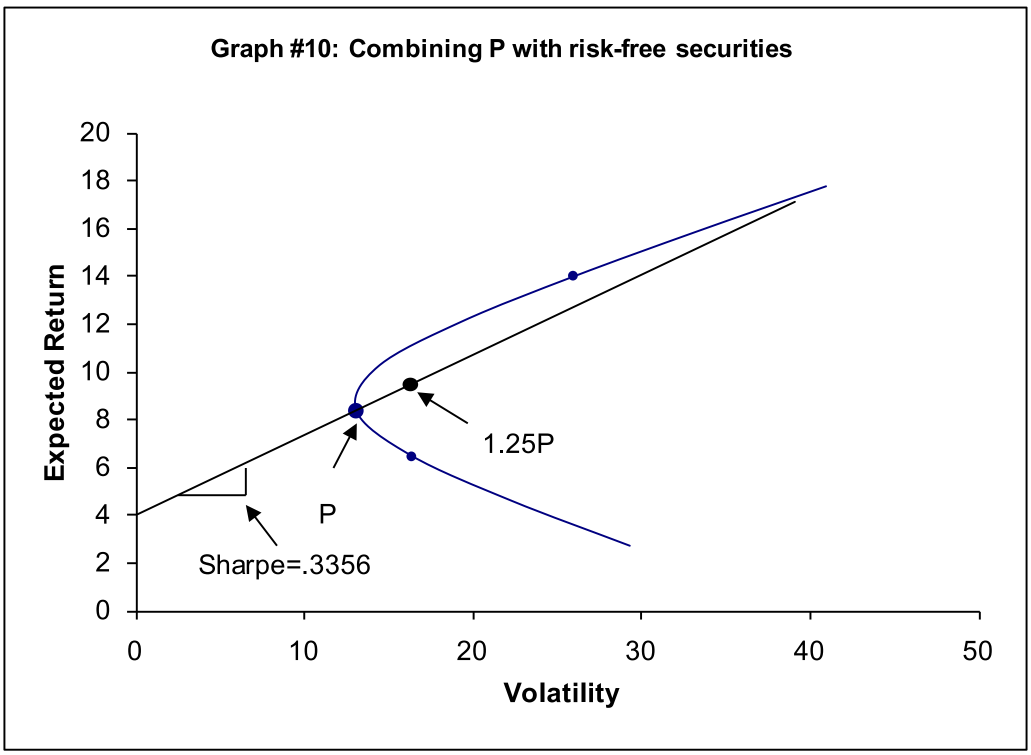
\$ invested in JPM and GD:

JPM = 375,000 =

GD = 125,000=

$SD(R_{1.25P}) = 16.3\% =$

$E(R_{1.25P}) = 9.47\% =$



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Q: Where is 2P? How would you build this portfolio?

Q: Can we do better than P?

Goal =>

=>

D. Identifying the Optimal Risky Portfolio

1. Sharpe Ratio $= \frac{E(R_P) - r_f}{SD(R_P)}$ (11.17)

=> slope of line that create when combine risk-free investment with risky P

Ex. Sharpe ratio when invest $300,000 in JPM and $100,000 in GD.

Sharpe Ratio = 0.3356 = $\frac{8.375-4}{13.04}$

=> see graph

Q: What happens to the Sharpe Ratio if choose a point just above P along curve?

=>

Q: What is "best" point on the curve?

2. Optimal Risky Portfolio

Key =>

Ex. Highest Sharpe ratio when x_{JPM} = .44722, x_{GD} = 1 – .44722 = .55278

Note: I solved for x w/ highest Sharp ratio using Solver in Excel

=> if invest $400,000 total, then invest $178,888 in JPM and $221,112 in GD

JPM = 178,888 =

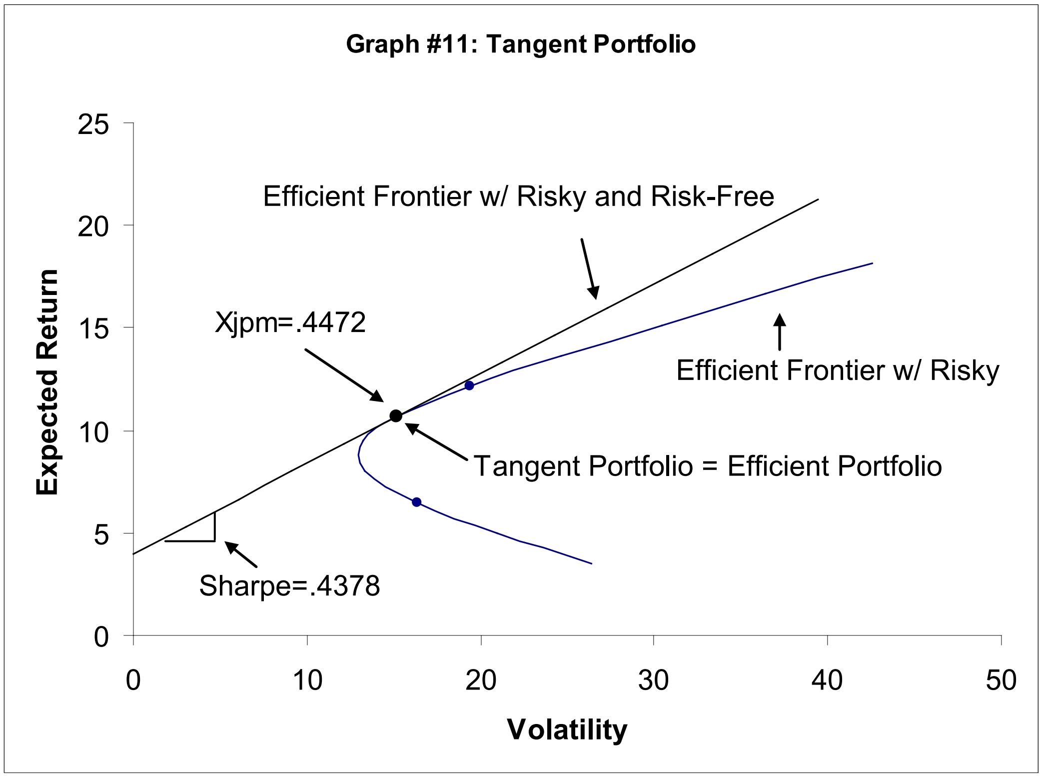
GD = 221,112 =

Note: $E(R_{JPM})$ = 6.5%, $E(R_{GD})$ = 14%; $SD(R_{JPM})$ = 16.3%, $SD(R_{GD})$ = 26%; and Corr (R_{JPM}, R_{GD}) = – 0.1382

$E(R_T)$ = 10.646% =

$$SD(R_T) = 15.182\% = \sqrt{(.44722)^2(16.3)^2 + (.55278)^2(26)^2 + 2(.44722)(.55278)(-0.1382)(16.3)(26)}$$

$$\text{Sharpe Ratio (Tangent)} = \frac{10.646-4}{15.182} = .4378 > .3356 = \textit{Sharpe Ratio } (P)$$



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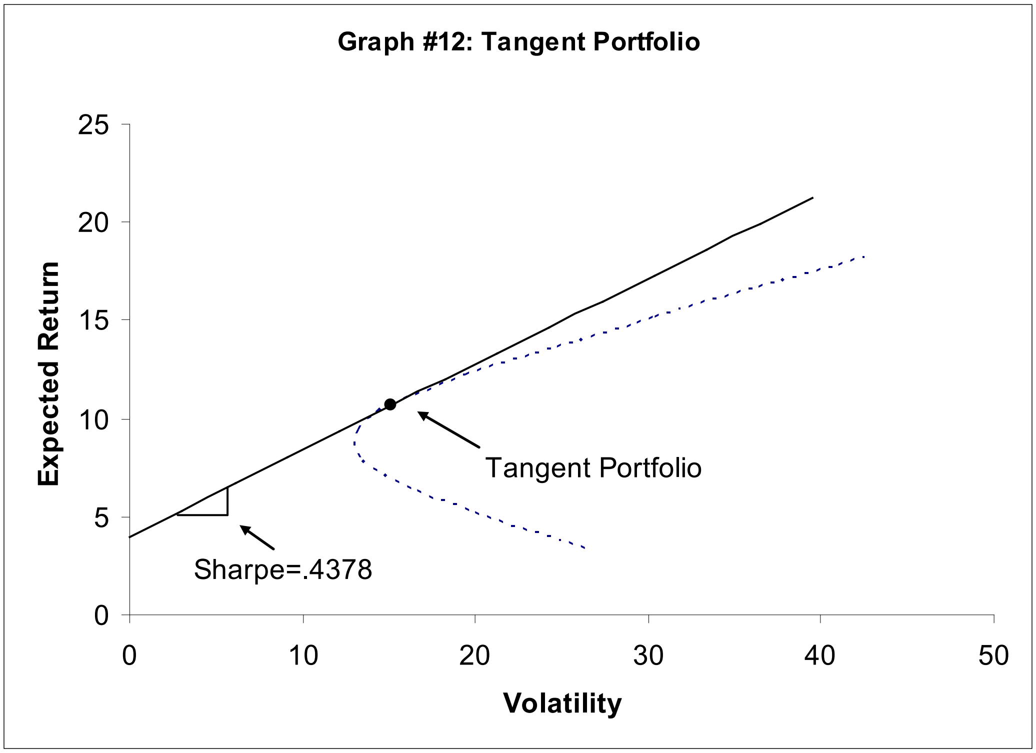
Implications:

1)

2)



=> show point if put .2 in Tangent Portfolio (TP), .9 in TP, and 1.5 in TP

IV. The Efficient Portfolio and Required Returns

A. Basic Idea

Q: Assume I own some portfolio P. Can I increase my portfolio's Sharpe ratio by short-selling risk-free securities and investing the proceeds in asset *i*?

A: I can if the extra return per unit of extra risk exceeds the Sharpe ratio of my current portfolio

Note: add graph to board that shows improving P by moving up and to right

1. Additional return if short-sell risk-free securities and invest proceeds in "i"

Use Eq. 11.3: $E[R_P] = \sum_i x_i E[R_i]$
=> $\Delta E(R_p) = \Delta x_i E[R_i] - \Delta x_i r_f = \Delta x_i (E[R_i] - r_f)$

2. Additional risk if short-sell risk-free securities and invest proceeds in "i"

Use Eq. 11.13 (from text): $SD(R_p) = \sum x_i Corr(R_i, R_p) SD(R_i)$

=> $\Delta SD(R_p) = \Delta x_i Corr(R_i, R_p) SD(R_i)$

3. Additional return per risk = $\frac{\Delta x_i(E[R_i]-r_f)}{\Delta x_i Corr(R_i,R_p)SD(R_i)} = \frac{E[R_i]-r_f}{Corr(R_i,R_p)SD(R_i)}$

4. Improving portfolio

=> I improve my portfolio by short-selling risk-free securities and investing the proceeds in "i" if:

$$\frac{E[R_i]-r_f}{Corr(R_i,R_p)SD(R_i)} > \frac{E[R_p]-r_f}{SD(R_p)}$$

Or (equivalently):

$$E[R_i]-r_f > SD(R_i) \times Corr(R_i,R_P) \times \frac{E(R_P)-r_f}{SD(R_P)} \tag{11.18}$$

B. Impact of people improving their portfolios

1. As I (and likely other people) start to buy asset i, two things happen

1)

2)

2. Opposite happens for any asset i for which 11.15 has < rather than >

C. Equilibrium

1) people will trade until 11.18 becomes an equality
2) when 11.18 is an equality, the portfolio is efficient and can't be improved by buying or selling any asset

$$E[R_i]-r_f = SD(R_i) \times Corr(R_i,R_{Eff}) \times \frac{E(R_{Eff})-r_f}{SD(R_{Eff})} \tag{11.A}$$

3) If rearrange 11.A and define a new term, the following must hold in equilibrium

$$E(R_i) = r_i \equiv r_f + \beta_i^{Eff} \times (E[R_{Eff}]-r_f) \tag{11.21}$$

where:

$$\beta_i^{Eff} = \frac{SD(R_i) \times Corr(R_i,R_{Eff})}{SD(R_{Eff})} \tag{11.B}$$

r_i = required return on i = expected return on i necessary to compensate for the risk the assets adds to the efficient portfolio

V. The Capital Asset Pricing Model

A. Assumptions (and where 1st made similar assumptions)

1. Investors can buy and sell all securities at competitive market prices (Ch 3)
2. Investors pay no taxes on investments (Ch 3)
3. Investors pay no transaction costs (Ch 3)
4. Investors can borrow and lend at the risk-free interest rate (Ch 3)
5. Investors hold only efficient portfolios of traded securities (Ch 11)
6. Investors have homogenous (same) expectations regarding the volatilities, correlations, and expected returns of securities (Ch 11)

Q: Why even study a model based on such unrealistic assumptions?

1)

=>

2)

=>

3)

B. The Capital Market Line

1. Basic idea:

Rationale:

1) By assumption, all investors have the same expectations

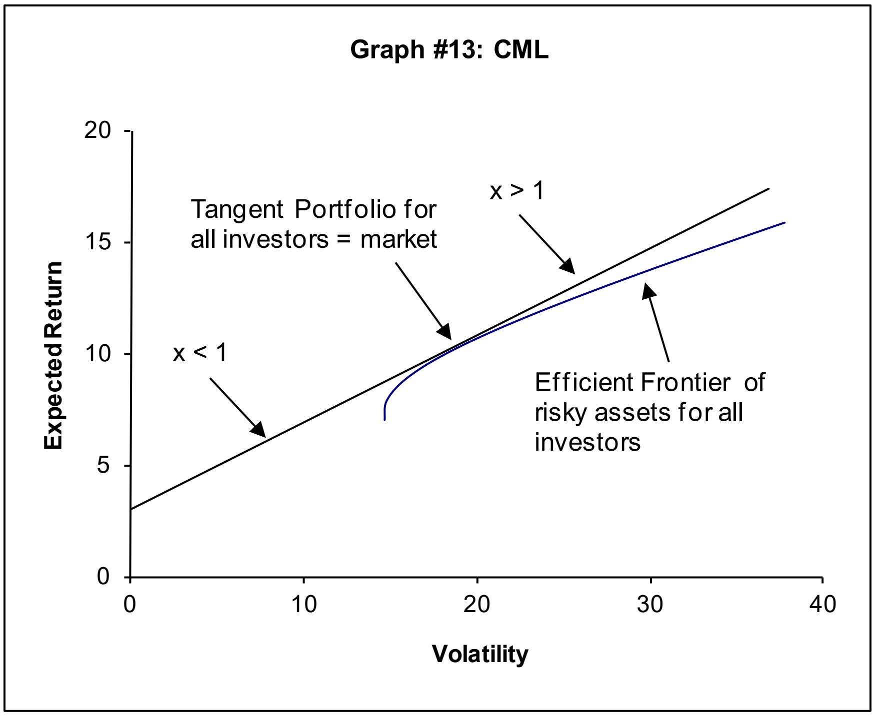
2)

3)

4)

5)

2. Capital Market Line: Optimal portfolios for all investors:



C. Market Risk and Beta

If the market portfolio is efficient, then the expected and required returns on any traded security are equal as follows:

$$E(R_i) = r_i = r_f + \beta_i \times \left(E[R_{Mkt}] - r_f\right) \tag{11.22}$$

where: $\beta_i = \beta_i^{Mkt} = \frac{SD(R_i) \times Corr(R_i, R_{Mkt})}{SD(R_{Mkt})} = \frac{Cov(R_i, R_{Mkt})}{Var(R_{Mkt})}$ (11.23)

Notes:

1) substituting β_i^{Mkt} for β_i^{Eff} and E[R_{Mkt}] for E[R_{Eff}] into 11.21
2) will use β_i rather than β_i^{Mkt}
3) rather than using equation 11.23, can estimate beta by regressing excess returns (actual returns minus risk-free rate) on security against excess returns on the market

=> beta is slope of regression line

Ex. Assume the following returns on JPM and the market. What is the beta of JPM? What is the expected and required return on JPM if the risk-free rate is 4% and the expected return on the market is 9%?

	Return on:	
Year	JPM	Market
1	-21%	-19%
2	7%	-2%
3	14%	17%
4	-3%	4%
5	23%	7%
6	19%	18%

$\bar{R}_{JPM} = 6.5$
$Var(R_{JPM}) = 266.3$
$SD(R_{JPM}) = 16.3$
=> see pages 3 and 4 for these calculations

$\beta_{JMP} = \frac{Cov(R_{JPM}, R_{Mkt})}{Var(R_{Mkt})}$

$Cov(R_{JPM}, R_{Mkt}) = \frac{1}{T-1}\sum_t (R_{JPM,t} - \bar{R}_{JPM})(R_{MKT,t} - \bar{R}_{Mkt})$

$\bar{R}_{Mkt} = 4.2 = \frac{1}{6}(-19 - 2 + 17 + 4 + 7 + 18)$

=> $Cov(RJPM, RMkt) = 190.3 = \frac{1}{5}\big((-21 - 6.5)(-19 - 4.2) + (7 - 6.5)(-2 - 4.2) + (14 - 6.5)(17 - 4.2) + (-3 - 6.5)(4 - 4.2) + (23 - 6.5)(7 - 4.2) + (19 - 6.5)(18 - 4.2)\big)$

$$Var(R_{Mkt}) = 187.7 = \frac{1}{5}[(-19 - 4.2)^2 + (-2 - 4.2)^2 + (17 - 4.2)^2 + (4 - 4.2)^2 + (7 - 4.2)^2 + (18 - 4.2)^2]$$

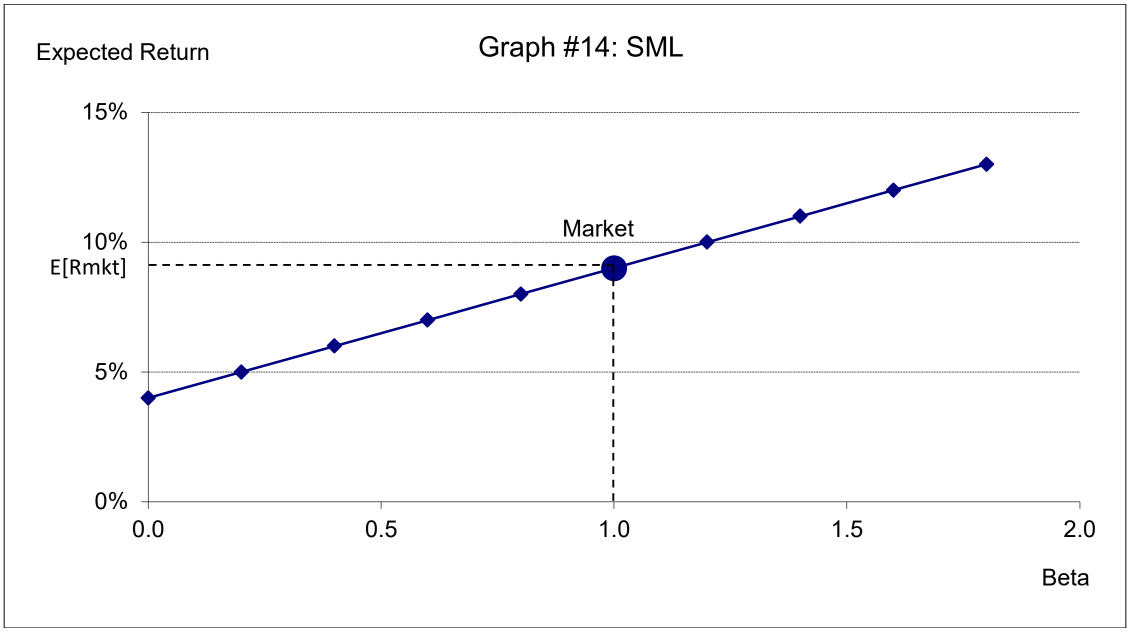
=> $\beta_{JMP} = 1.013 =$

$E(R_{JPM}) = \boldsymbol{r_{JPM}}$ =9.065%=

D. The Security Market Line (SML)

1. Definition: graph of equation 11.22: $E(R_i) = r_i = r_f + \beta_i \times \left(E[R_{Mkt}] - r_f\right)$

=> linear relationship between beta and expected (and required) return



Video

2. All securities must lie on the SML

=> expected return equals the required return for all securities

Reason:

=>

=>

=> JPM will lie on the SML just above and to the right of the market

3. Betas of portfolios

$\beta_P = \sum_i x_i \beta_i$ (11.24)

Note: see Equation (11.10) on separating out $\sum_i x_i$

Ex. Assume beta for JPM is 1.013 and that beta for GD is 0.159. What is beta of portfolio where invest $300,000 in JPM and $100,000 in GD?

$x_{JPM} = .75$, $x_{GD} = .25$

$\Rightarrow \beta_P = .7995 =$