

Chapter 4: The Time Value of Money

Big Picture: Cash flows at different points of time are in different units

=> can't compare or combine unless convert to the same units (same point in time)

I. Solving time value of money problems

A. Systematic steps

1. Draw a timeline
2. Identify the pattern of the cash flow:
 - Perpetuity – **series of regular cash flows that continue forever**
 - Annuity – **series of regular cash flows that stop**
 - Single cash flow – **one cash flow**
3. Write down appropriate equation, plug in what know, solve for any one variable don't know
4. Repeat steps 2 and 3 as often as necessary to solve the problem

B. The Timeline

1. I do all time lines in years
2. I vary size of tick marks to make timelines easier to read

Relative size of tick marks (from smallest to largest): monthly, semiannual, annual, 5-year, 10-year

Note: use as few “sizes” as possible

3. You can click on most “timelines” to access a video clip of the timeline

C. Basic Time Value of Money Equations

1. Single Cash Flow

a. Future Value: $FV_t = C \times (1 + r)^n$ (4.1)

b. Present Value: $PV_t = \frac{C_n}{(1+r)^n} = C_n \left(\frac{1}{1+r}\right)^n = C \times (1 + r)^{-n}$ (4.2)

2. Perpetuity (Present Value Only)

$$PV_t = \frac{C}{r - g} \quad (4.10)$$

Note: if constant cash flow, $g = 0$.

3. Annuity

$$\text{a. Present Value: } PV_t = \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r} \right)^N \right) \quad (4.11)$$

$$\text{b. Future Value: } FV_t = \frac{C}{r-g} \left((1+r)^N - (1+g)^N \right) \quad (4.A)$$

D. General notes on TVM equations

- 1) $C = 1^{\text{st}}$ cash flow in series (may be only cash flow)
- 2) $r =$ interest rate
- 3) $g =$ growth rate of cash flows (after 1st one) in a series of cash flows
Note: $g = 0$ for constant cash flows and $g < 0$ for shrinking cash flows
- 4) $n =$ number of periods moving a cash flow into the future or towards the present
- 5) $N =$ number of cash flows in a series of cash flows
- 6) Where end up on time-line:
 - a) PV of single cash flows: **n periods before the cash flow**
 - b) FV of single cash flows: **n periods after the cash flow**
 - c) PV of a series (growing annuity or growing perpetuity): **one period before first cash flow**
Note: **period defined by time between cash flows**
 - d) FV of series (growing annuity): **date of last cash flow**
- 7) I add a “t” subscript on PV and FV to keep track of where I end up on the timeline after each step

II. Examples

A. General Examples

1. Upon retirement 35 years from today, you would like to make your first of 15 annual withdrawals from your savings account. You would like for the first withdrawal to be \$180,000 and would like to be able to increase your withdrawals by 5% per year in order to allow for inflation. How much would you have to deposit today to achieve your goal if your account pays 8.5% per year? What is the size of your second withdrawal and your final withdrawal?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

Deposit today:

$$PV_t = \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r} \right)^N \right)$$

$$PV_{34} = 1,998,022.07 = \frac{180,000}{.085-.05} \left(1 - \left(\frac{1.05}{1.085} \right)^{15} \right)$$

Steps: 2) pattern, 3) equation, 4) fill in

$$PV_t = \frac{C_n}{(1+r)^n}$$

$$PV_0 = 124,735.24 = \frac{1,998,022.07}{(1.085)^{34}}$$

Second withdrawal:

Steps: 1) [timeline](#), 2) pattern, 3) equation, 4) fill in

$$FV_t = C \times (1+r)^n$$

$$C_{36} = 189,000 = 180,000(1.05)^1$$

Final withdrawal:

Steps: 1) [timeline](#), 2) pattern, 3) equation Q: PV or FV?

$$FV_t = C \times (1+r)^n$$

$$C_{49} = 356,387.69 = 180,000(1.05)^{14}$$

2. Assume you plan to deposit \$100 into a savings account 5 years from today. Thereafter you plan to make annual deposits that grow by 3% per year. How much is in your account after you have made 7 total deposits if the account pays 9% per year? What is size of 7th deposit?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

$$FV_t = \frac{C}{r-g} \left((1+r)^N - (1+g)^N \right)$$

$$FV_{11} = 996.94 = \frac{100}{.09-.03} \left((1.09)^7 - (1.03)^7 \right)$$

Steps: 2) pattern of cash flow, 3) write down equation; Q: PV or FV?

$$FV_t = C \times (1+r)^n$$

$$CF_{11} = 119.41 = 100(1.03)^6$$

3. Assume you deposit \$1000 in an account that earns an interest rate of 6% per year. You plan to make annual withdrawals from your account beginning four years from today. Your plan for your first withdrawal to equal \$59.55 and for your subsequent withdrawals to continue forever. At what rate can your withdrawals grow?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

$$FV_t = C \times (1+r)^n$$

$$FV_3 = 1191.02 = 1000(1.06)^3$$

$$PV_t = \frac{C}{r-g}$$

$$1191.02 = \frac{59.55}{.06-g}$$

$$\Rightarrow g = .01$$

4. Assume that you have just deposited \$500 into an account that pays an interest rate of 8% per year. You plan to make equal annual withdrawals from this account with the first withdrawal coming 5 months from today and the final withdrawal coming 5 years and 5 months from today. How large can you make each of your withdrawals?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

$$PV_t = \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r} \right)^N \right)$$

$$PV_t = 478.05 = \frac{c}{.08} \left(1 - \left(\frac{1}{1.08} \right)^6 \right)$$

Q: Where end up on the timeline when take present value of the withdrawals?

=> 7 months ago

=> need to figure out equivalent 7 months ago of \$500 today (so can solve for C)

Steps: 2) pattern, equation; Q: PV or FV?

$$PV_t = \frac{c_n}{(1+r)^n}$$

$$PV_{-7mo} = 478.05 = \frac{500}{(1.08)^{7/12}}$$

$$PV_t = \frac{c}{r-g} \left(1 - \left(\frac{1+g}{1+r} \right)^N \right)$$

$$PV_{-7mo} = 478.05 = \frac{c}{.08} \left(1 - \frac{1}{(1.08)^6} \right) \Rightarrow C = 103.41$$

B. Practicing “n” and “N”

1. Assume you plan to deposit \$100 per year into an account. You will make your first deposit five years from today and your final deposit eleven years from today.

a. Q: Assume you want to calculate the balance in your account right after your final deposit. Which equation would you use? What would you use for “N” or “n”?

Steps: 1) [draw \(or modify\) timeline](#), 2) identify pattern of cash flow, 3) figure out if need to calculate present or future value of payment(s), 4) identify equation or equations need to use, 5) figure out “N” or “n”.

A: Equation = future value of an annuity (4.A); $N = 7$

b. Q: Assume you that after making your final deposit, you leave the account alone until 15 years from today. What equation would you use and what would you use for “N” or “n” when calculating the balance of your account?

Steps: 1) [draw \(modify\) timeline](#), 2) identify pattern of cash flow, 3) figure out if need to calculate present or future value of payment(s), 4) identify equation or equations need to use, 5) figure out “N” or “n”.

A: Equation = future value of a single sum (4.1); $n = 4$

2. Assume you had planned to deposit \$100 per year into an account with the first deposit being made five years from today and the final deposit occurring eleven years from today. However, you have now decided to simply deposit an equivalent into an account today.

a. Q: What equation would you use to calculate the present value of the payments? What would you use for “N” or “n”? Where do you end up on the time line?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.

A: Equation = present value of an annuity (4.11); $N = 7$; end up at $t = 4$.

b. Q: Once you solve for the value at $t = 4$, how would you calculate the equivalent present value today (equation, N/n)?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.

A: Equation = present value of single cash flow (4.2), $n = 4$

3. Assume you have just deposited \$1000 into a savings account. You plan to make a series of annual withdrawals from the account starting eight years from today and continuing through 18 years from today.

Q: What equations will you need to use to solve for the amount you can withdraw? What will you use for N or n in each equation?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out "N" or "n".

A: 1) future value of single cash flow (4.1), $n = 7$; 2) present value of an annuity (4.11), $N = 11$

4. You have just made the first of a series of annual deposits of \$100 each into a savings account. You plan to make additional deposits through four years from today. Ten years from today, you plan to make the first of a series of annual withdrawals from your account. Your final withdrawal will occur 18 years from today.

a. Q: What equations will you need to use to solve for the amount of your first withdrawal? What would you use for N or n in each equation?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out "N" or "n".

A: 1) future value of annuity (4.A), $N = 5$; 2) future value of a single cash flow, $n = 5$; 3) present value of an annuity (4.11), $N = 9$

b. Q: What equations will you need to use to solve for the amount of your first withdrawal if you first calculate the present value of the deposits and solve for the future value of the withdrawals?

Steps: 1) [timeline](#), 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out "N" or "n".

A: 1) present value of annuity (4.11), $N = 5$; 2) future value of a single cash flow, $n = 19$; 3) future value of annuity, $N = 9$