Lecture 1: what's economics?

This class is for most of you your first introduction to the math foundations of microeconomics. So, to help you gain a bird's eyes view of microeconomics, I will first describe some concepts & terms.

A. models.

Microeconomics is a collection of mathematically based models that describe narrow features of human behavior. Not exhaustive depictions — this is not a replacement of psychology, sociology, or any of the social science branches you can name. It's a narrow modeling, and I will explain to you its distinctiveness & explore it with you in the context of these models.

Models are unique objects. Depending on your background, & exposure to much science, you may find models to be very unusual. For instance, I may say things like "assuming people have transitive, continuous & complete preference orderings, then this model predicts they will buy more of something when its price rises so long as the substitution effect is smaller than the income effect." It's a strange way of talking, first of all, but even stranger maybe is the idea that I am making prediction about what a person will do in some situation all because I solved ~~a statistical~~ a simple constrained optimization problem — when I know "real people" don't solve optimization problems when they buy things. So part of this class is to teach you about the usefulness of theoretical models, and the usefulness of stories, even when those stories aren't exactly "true", per se.

Models are like maps. They are like them in purely a metaphorical way, I mean. Say that you go to Florence, Italy — an ancient city filled with history & culture. Maps are useful to you if you've never been there, but only if you have a certain purpose. If your goal is to see all the cathedrals, for instance, then a spatial map showing the locations is about what you need. But what if you are wanting a history tour experience? What if you are an engineer ~~studying~~ needing to work on ~~~~ rebuilding its ~~~~ piping & water supply? ~~~~ Then a map of the cathedrals aren't so helpful. You might need a different map then — one perhaps providing information about building structures, ground water, piping & its age... It may not need to ~~~~ show or even name the churches at all.

The map metaphor is helpful, so I want to say one more thing before we move on. It isn't just that maps should be defined differently for different purposes, though that's true. It's also that there is no one map that is ever "true," or the real map. All maps are wrong, too, but not all maps are useless. Some maps are ~~useless~~ useful even when those maps are blatantly "wrong." Like maybe a map of ~~SF~~ Africa prior to Europeans began colonizing it. Maybe all the details are we now know wrong so it's not useful now for exploration, but perhaps it becomes useful to a historian.

But, even Google maps for SF, arguably a much more accurate map, isn't "true." For one it leaves out so much information, even though it's such an efficient ~~infor~~ map at displaying info. We don't know everything about SF from this map. And even stuff included is just an abstraction of the real thing.

I stress this b/c I want to say that theoretical models are not the real thing. ~~But in a way~~ ~~So~~ But ~~the fact~~ not b/c they are mathematical — all theories of the world are abstractions, whether the theory is expressed mathematically, ~~or~~ in words, or in diagrams. And no layering on of more info will ever change that.

Economic models are theoretical ~~of~~ objects that "map" onto the "real world"; but there are many diff. models attempting that. So how do we choose b/w them? How do we decide whether model A or B is correct if all models are fundamentally incorrect?

Economic models therefore ~~need~~ must satisfy two separate lines of scientific validity. ~~and probab~~ First, the models must be logically or/and mathematically correct. So most of our class focuses on that; we plumb the depths of seemingly simple models to determine what we can say or conclude and the underlying assumptions/conditions/premises for those declarative statements. If a statement or a theorem is based on a math error, then clearly that's "wrong". The conclusion isn't supported by the premises. So a great deal of the mathematical framework is a kind of internal validation & careful studying of models & their associated conclusions like a syllogism. Where we think a theory's ~~model's~~ conclusion is incorrect, we must first determine if the model guaranteed that at all, & if so, why?

This kind of logical or mathematical verification of a theory is common in the history of science, and particularly in those fields where empiricism ~~was~~ to experimentation seemed to come late (like economics) or not at all (theoretical physics).

The second line of verification comes from the empirical tradition. ~~Popper~~ Karl Popper in particular is identified with this tradition. Put succinctly, a scientific theory must do more than just try to explain. ~~Theor~~ Scientific theories must, according to this later tradition, be "falsifiable". A theory is falsifiable iff the theory makes ~~predictions~~ causal predictions. Ideally, these causal predictions would be also observable, but sometimes the causal prediction may be an outcome which are unobservable to the scientist. The theory that a large comet struck Earth several hundred million years ago to destroyed the dinosaur population, for instance, is maybe a difficult theory to test because at first glance, it's not clear how we would falsify that. Stop comets from striking Earth to see if dinosaurs survived anyway? ~~Or maybe that~~

Sometimes we find data that ~~creates~~ correlates with the models predictions, but is this an actual falsification of the theory? Say that my theory was that ~~the~~ college education caused worker productivity to rise & therefore in equilibrium, caused their wages to rise. That is an empirical causal prediction:

- if worker A goes to college, she gets W_A^C.
- if worker A DID NOT go to college, she gets W_A^{NC}.

① $\longrightarrow W_A^C > W_A^{NC}$

- So, just look at worker A both & see, right? Or maybe look at workers in college & workers not in college?

IS: $W_A^{college} \gtrless W_A^{no\,college}$

the same as

② $\longrightarrow W_A^{College} \gtrless W_B^{No\,college}$

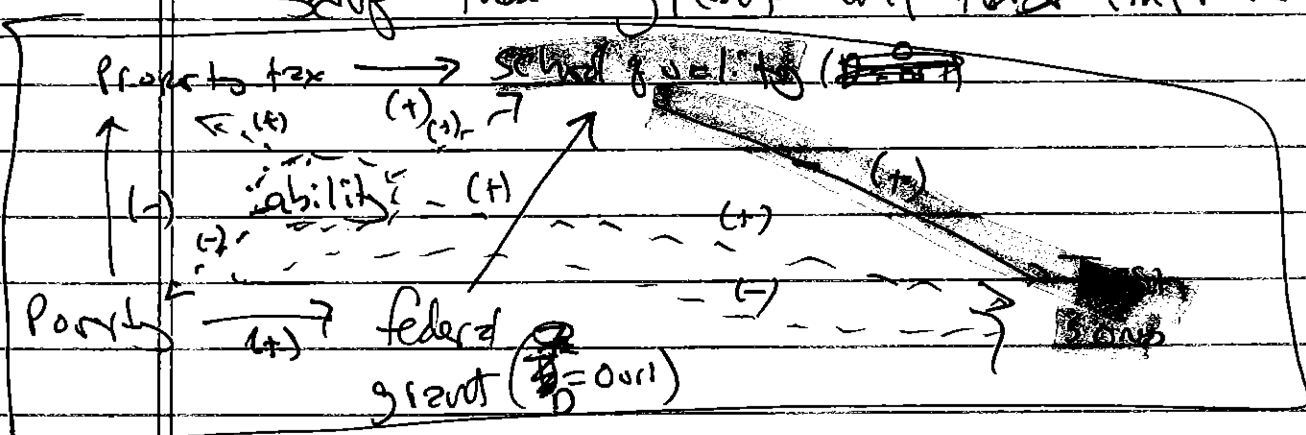
- No, it isn't. This may seem overly technical, but it is critically important, so hear me out. A scientific theory makes ~~predictions~~ causal predictions, not correlational predictions. That means that falsification ~~most~~ is associated w/ testing equation 1, not 2.

Empirical falsification of a theoretical "causal prediction" maps back to theoretical causal prediction itself. Economic models make very ~~simple~~ causal predictions about simple relationships. Like super simple. Like "Price causes Quantity Demanded to fall" or "wages cause cost to rise". Like simple relationships. It's a very step-by-step approach, i.w. very deliberate. But more than that, it is an approach that is simple b/c it recognizes that there are multiple causes of some outcome, but ~~the~~ theoretical causal prediction will be assuming those other causes are fixed, or unchanging, so that we can just focus on one mechanism or a chain of mechanisms at a time. This is the "ceteris paribus" assumption — holding all things constant.

But, remember in a theoretical, mathematically-based modelling framework, we have this luxury. For example, we will do a lot ~~of~~ with a little multivariate calculus, & those techniques explicitly "hold constant" other dimensions, or variables, in the model. It's clean, logical, & precise. So when I have a model which says "an increase in schooling will increase wages", that model held constant the causal effect of motivation, IQ, parental education, peer pressure, awesome teachers, ~~[illegible]~~ race & so forth on wages.

But in the real world, when I just ~~take~~ scoop out 1,000 workers ~~from~~ w/ college education & 1,000 w/out, none of these things held constant in the data in fact are held constant.

Consider the following scenario. Let's say that Poverty causes ~~nutrition to be worse~~, causes schooling to be worse due to property taxes used to pay teacher salaries & school materials. Let's also assume ~~poverty~~ federal government grants go to areas with ~~more poverty &~~ worse performing schools ~~nutrition~~. Finally, let's say that ~~nutrition~~ ~~grants~~ we ~~grants go to improve~~ have a theory which says that grants will improve test scores.



* Causal prediction is the pink arrow. It is "increase ~~school quality~~ federal grants & you will increase test scores — holding all determinants of test scores, like ability & poverty, constant, Z." (+)

* But notice that if we compare ~~grants~~ places with the grants (D=1) to those without (D=0), we are

not comparing apples to apples, but apples to oranges.
Why?

→ Grants turn on "in the real world"
b/c of poverty.
↳ Poverty, though, reduces property taxes (−).
↳ Property taxes ~~reduce~~ increase school quality (+)
↳ Poverty is the consequence of low ability people concentrated in an area which depresses property taxes and lowers school quality AND lowers test scores.

- So our theory predicts if you could just take a lever to turn grants on to off, but hold poverty, etc., the same, THEN you would see a positive correlation b/w grants to scores. But, that thought experiment is based on ~~the~~ what is happening in the model; the real world, grants are increasing or falling for reasons that aren't "ceteris paribus", to those things changing ~~may~~ can create ~~...~~ spurious correlation.

The challenge for you as young economists is to become competent in both of these forms of validation/verification. ~~...~~ And our class this semester is about the inner logic for theory theoretical predictions assoc. w/ studying foundational models about households & firms, & then use them to understand better what econ. teaches us about the world/society, as well as what it does not. This class is about learning to think like an economist, which means learning to build & learn from simple, yet powerful, models. Your econometrics classes w/ Jim West & myself ("Causal Inference") are about the latter methods of validation & falsification.

Before I conclude this overview of ~~what~~ what a model is scientifically, I wanted to say one last thing about the generality of a model & that is w/ regards to "causal predictions" versus "ethical acceptance/rejection of some aspect of the real world contained in the model".

Let's use a simple example. Let's say I have a theory that some disease (like cholera) is transmitted by water. So, contaminated drinking water for instance will cause someone to become choleric, & possibly die. That is a causal prediction based on some biological model of the disease. Presumably that's true or false, too, regardless of ~~the~~ whether the water supplies are more likely contaminated in ~~areas~~ poorer places, & regardless of whether the providers of clean water in an area is a corporation that you love/despise b/c of their ethical/business practices.

Economic models make other, more subtle, distinctions b/w scientific ~~[illegible]~~ content and ethical ~~[illegible]~~ judgment. A ~~[illegible]~~ causal prediction would be "if you increase school quality, you'll increase worker wages." Should we, though, do that? That is, just b/c something like a causal prediction is true, does that mean therefore that ~~[illegible]~~ we should do it? ~~[illegible]~~

Notice the subtle difference in language. Scientific theories in their causal predictions are not evaluative judgments - they're simple predictive relationships from an underlying causal model. So in principle, there shouldn't be disagreement b/w two people w/ opposing political views who have the same information about some model's behavioral predictions. But when statements shift from "what is" to "what should someone do", we might expect more disagreement. Just b/c we think the abundance of evidence says min. wage laws will decrease employment & raise unemployment among lowskill workers, that does not mean that economists agree about whether that policy should be enacted or not.

this distinction between "what is" & "what should be" is key. You hear it more often called the "positive/normative distinction". Positive theories are ~~simple~~ models that make falsifiable claims. Normative theories focus on what we should do. While normative theory does enter into our discussions this semester, most of the time we will be in the world of "positive economic theory".

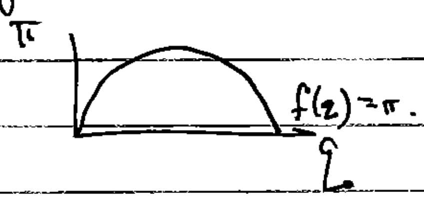
Math Review - I am now going to shift gears a little bit so that I can review some of the basic mathematics we will use this semester. I will now review optimization principles & the calculus that I need you to learn. Most of this will be focused on techniques to find the optimal value of some function. Forgive me if this is too simple.

I. Maximization of a function of one variable.

ex: Say a manager wishes to maximize profits which are only a function of how much ~~she~~ quantity she sells. Mathematically,

$$1. \quad \pi(q) = f(q)$$

Let's say that there's a rel. b/w π & q like this:



As $q \uparrow$, the value of the profit fn. changes, as does the direction of that.

9. Rules for finding deriv.

1. Let b is a constant.

$$\frac{db}{dx} = 0.$$

2. Let ~~modify~~ pre-mult. a fn., f.

$$y = bf(x)$$

$$\frac{dy}{dx} = \frac{d(bf[x])}{dx} = b \cdot \frac{df(x)}{dx} = bf'(x)$$

3. $y = x^b$. b is the expont.

$$\frac{dy}{dx} = \frac{d(x^b)}{dx} = bx^{b-1}$$

ex: x^2

$$\frac{dx^2}{dx} = 2x^{2-1} = 2x$$

4. natural log of x. $y = \ln x$.

$$\frac{dy}{dx} = \frac{d \ln x}{dx} = \frac{1}{x}$$

5. Let $y = a^x$, where a is constant.

$$\frac{dy}{dx} = \frac{d(a^x)}{dx} = a^x \ln a$$

5'. Special case. if $y = e^x$

$$\frac{dy}{dx} = \frac{de^x}{dx} = e^x$$

6. Assume $f(x)$ and $g(x)$ are two fn. & their deriv. exist.

Let $y = f(x) + g(x)$ (Addition Rule)

$$\frac{dy}{dx} = \frac{d\,[f(x) + g(x)]}{dx}$$

$$= \frac{d\,[f(x)]}{dx} + \frac{d\,[g(x)]}{dx}$$

$$= f'(x) + g'(x)$$

7. Let $y = f(x) \cdot g(x)$ (Mult. rule)

$$\frac{dy}{dx} = \frac{d\,[f(x)g(x)]}{dx}$$

$$= f'(x)g(x) + f(x)g'(x)$$

8. Let $y = \frac{f(x)}{g(x)}$ (Quotient rule)

$\frac{T'B - B'T}{B^2}$ if $g(x) \neq 0$

$$\frac{dy}{dx} = \frac{T'B - B'T}{B^2}$$

$$\left.\begin{array}{l} T' = f'(x) \\ B' = g'(x) \\ B^2 = g(x)^2 \end{array}\right\} \rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

9. Let $y = f(x)$ And f' & g' exist.
$x = g(z)$

$$\frac{dy}{dz} = \frac{d}{dz}\left(f(g(z))\right)$$

$$= \frac{dy}{dx} \cdot \frac{dx}{dz}$$

$$= \frac{df}{dx} \cdot \frac{dg}{dz}$$

$$\frac{dy}{dz} = f'g' \qquad \text{(Chain Rule)}$$

~~max in Multiple Vars~~

ex: Find ~~max~~ optima of $y = 3x^2 + 12x + 1$
Check if it is max or min using FOC & SOC.

FOC: $\frac{dy}{dx} = 6x + 12 = 0$

$$\frac{6x}{6} = \frac{-12}{6}$$

$$\boxed{x = -2}$$

SOC: $\frac{d^2y}{dx} = 6 > 0$ If SOC > 0, it's a min.
If SOC < 0, it's a max.

Show online using google

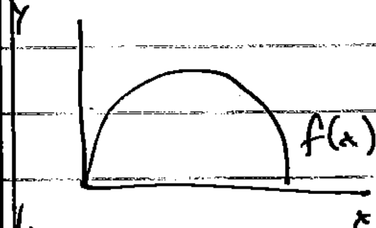
Maximization w/ one vbl.

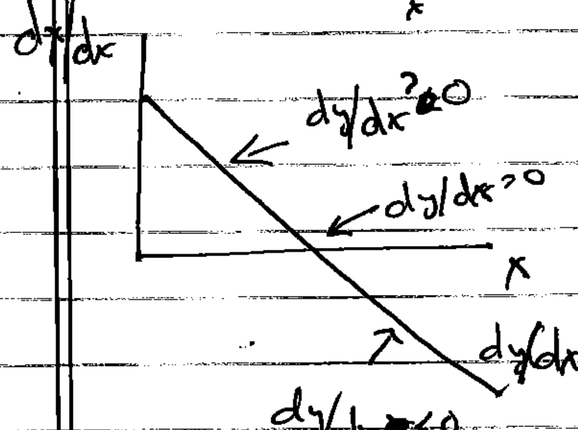
let $y = f(x)$

$\frac{dy}{dx}$ ~ derivative of y wrt x. Measures the rate of change in the y vbl as x changes by some infinitesimal amt.

↳ Also $f'(x)$

. Measures the slope of the y fn. at some point on the fn. ~~[illegible]~~ equalling x.





← Notice that the slope's value is decreasing as x increases though,

$\left(\frac{d}{dx}\right)\left(\frac{dy}{dx}\right)$: second derivative : $f''(x)$

$$\frac{d^2y}{dx^2} < 0$$

ex: $y = 3x^2 + 12x + 1$

$\frac{dy}{dx} = 6x + 12$

Set $dy/dx = 0$. Solve for x.

$\frac{dy}{dx} = 0$, $6x + 12 = 0$

$\frac{6x}{6} = \frac{-12}{6}$

$\boxed{x = -2}$

max or min ↳ $\frac{d^2y}{dx^2} = 6$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x+12) = 6$

-2 is the optimum. Is it max or min?

(5)

ex: Let π (profit) equal:

$$\pi = 1000q - 5q^2$$

FOC. $\frac{d\pi}{dq} = 1000 - 10q = 0$

$$10q = 1000$$
$$\frac{10q}{10} = \frac{1000}{10}$$
$$\boxed{q^* = 100}$$

SOC $\frac{d^2\pi}{dq^2} = -10 < 0$ (max)

global max b/c second deriv is always -10. Therefore $q^* = 100$ is global max.

Opt. of Mult. Var. fn.

Partial deriv. (8)

Suppose health y is some fn. of carbs (c), protein (P), + fats (F):

$$y = f(C, P, F)$$

We wonder what happens to her health if she increases her cons. of carbs, but holding constant P+F.

So $\frac{\partial y}{\partial c} = \frac{\partial f}{\partial c} = f_c(C, P, F)$ or $f_1(C, P, F)$

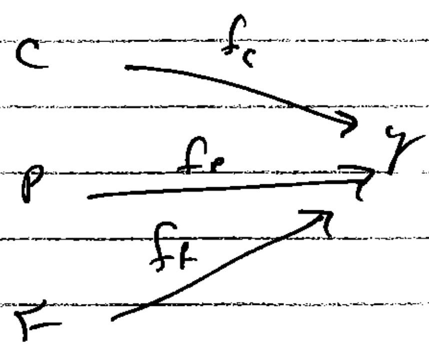
Calculating partials (Schotter)

ex 2: let $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$

$$\frac{\partial y}{\partial x_1} = 2ax_1 + bx_2$$

$$\frac{\partial y}{\partial x_2} = bx_1 + 2cx_2$$

Partials are math. expression of ceteris paribus.

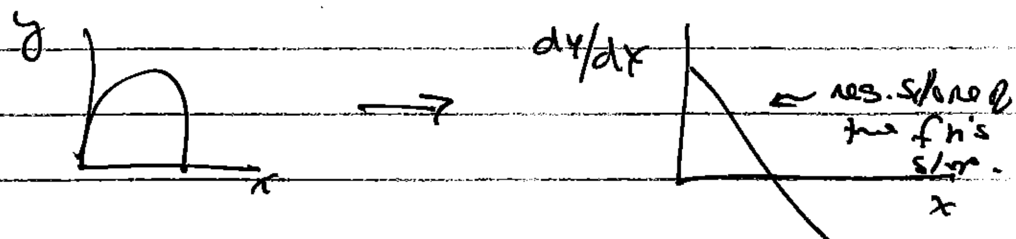


- Each causes y.
- Partial focuses only on one arrow.

$\frac{\partial y}{\partial C} = \frac{\partial f}{\partial C}$ would be top arrow

Second order partial derivatives

What about ~~ø~~ the partial of a partial deriv?
Recall when we did this with a ~~one~~. Second deriv in one variable cases. ~~Here~~, then it was easy:



it ~~was~~ is slightly more complex w/ 2 or more vbls

1. $y = (x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$

first partials

$$f_1 = \frac{\partial y}{\partial x_1} = 2ax_1 + bx_2$$

$$f_2 = \frac{\partial y}{\partial x_2} = 2cx_2 + bx_1$$

second partials Note that there are four second partials, not 2:

$$f_{11} = 2a$$

$$f_{12} = b$$

$$f_{21} = b$$

Notice: $f_{12} = f_{21}$ coincidence? No. See below.

$$f_{22} = 2c$$

Young's theorem - The order in which partial diff. is conducted to evaluate second-order partials doesn't matter:

$f_{ij} = f_{ji}$ for any pair of vbls x_i + x_j.

Young's theorem also implies that the matrix of all second-order partials of a fn. is symmetric.

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

the upper right is a mirror image of the lower left.

Total differential

Say that all the x's in fn., y, were allowed to change by a small amount. Then the TOTAL CHANGE in y ("total differential") is just the linear combination, or sum, of each partial derivative multiplied by the change in that variable, x.

Let $y = f(x_1, x_2, \ldots, x_n)$ for n variables.

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n.$$

Direct vs. indirect effects.

~~Let x_3~~

Let $y = f(x_1(a), x_2(a), \cancel{x_3}\, a)$

$$\frac{dy}{da} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial a} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial a} + \frac{\partial f}{\partial a}$$

$$= \underbrace{f_1 \frac{\partial x_1}{\partial a} + f_2 \frac{\partial x_2}{\partial a}}_{\text{indirect effect of a any}} + \underbrace{f_a}_{\text{direct effect.}}$$

Implicit Fn.

- if the value of a fn. is held constant, then the independent vbls can no longer take on any value. Rather they can take on only those set of values that result in the fn. retaining the req. value.
- called "implicit" b/c when you set the fn. equal to some value, an "implicit rel. is created among the independent variables entering into the fn.

ex: $x + 2y = 4$

$\frac{2y}{2} = \frac{4-x}{2}$

$\boxed{y = 2 - \frac{1}{2}x}$

Notice, diff. x values yield diff. value of y. B/c we did not set y to any particular value. But, had we set y to a particular value, we are setting x to be only one value.

↓

let $y = 0$.

$y = 2 - \frac{1}{2}x = 0$

$2 - \frac{1}{2}x = 0$

$\frac{1}{2}x = 2$

$\boxed{x = 4}$

if $y = 0$, x must equal 4.

another ex., this time w/ multipliers,

Let $y = f(x_1, x_2)$

Take total diff of y.

$$\boxed{dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2}$$

Now set y to 0. dy, therefore, is 0.

$$dy = 0$$

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

$$f_1 dx_1 + f_2 dx_2 = 0$$
$$f_1 dx_1 = -f_2 dx_2$$
$$\boxed{-\frac{f_1}{f_2} = \frac{dx_2}{dx_1}}$$

if $dy = 0$, then we necessarily are requiring x_1 to x_2 to have a particular rate of exchange. ~~dx~~ We are requiring a "trade-off" equalling, here, the ratio of f_1 to f_2 times -1.

Suppose that an agent wishes to max a multivariate fn, $y = f(x_1, \dots, x_n)$

Q1: What would the total change in y be from a change in x_1?

$$\frac{dy}{dx_1} = \frac{\partial f(x_1, \dots, x_n)}{\partial x_1}$$

$$\boxed{dy = f_1 \, dx_1}$$

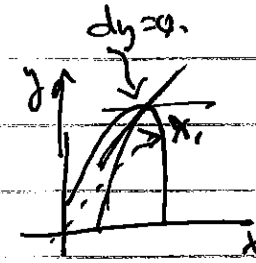
it'd be the rate of ch

X

how much

Q2: What are FOC + SOC for max when multivariate fn?

Necessary conditions are $dy = 0$.



But note, there are 2 ways to induce changes in y:
- thru X_1 + thru X_2
- therefore, we are necessarily requiring that the partials in both directions be set to zero.

FOC: Necessary condition for a max of fn $f(x_1, \dots, x_n)$ is that $dy = 0$ for any combo. of small ch. in x's

$$f_1 = f_2 = \cdots = f_n = 0$$

SOC: $f_{ii} < 0$.

Second partial derivatives must be negative to ensure the max.

ex: $y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$

$$y = -(x_1 - 1)(x_1 - 1) - (x_2 - 2)(x_2 - 2) + 10$$
$$y = -(x_1^2 - 2x_1 + 1) - (x_2^2 - 4x_2 + 4) + 10$$
$$y = -x_1^2 + 2x_1 - 1 - x_2^2 + 4x_2 - 4 + 10$$
$$y = -x_1^2 + 5 + 2x_1 - x_2^2 + 4x_2$$

$$y = -x_1^2 + 2x_1 + 4x_2 - x_2^2 + 5$$

Goal: max y choosing optimal x_1^* & x_2^*.

FOC 1: $\frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0$

$$\frac{2x_1}{2} = \frac{2}{2}$$

$$\boxed{x_1^* = 1}$$

FOC 2: $\frac{\partial y}{\partial x_2} = 4 - 2x_2 = 0$

$$\frac{4}{2} = \frac{2x_2}{2}$$

$$\boxed{x_2^* = 2}$$

$\boxed{x_1^* = 1;\ x_2^* = 2}$ to optimize y.

$f_{11} = \frac{\partial^2 y}{\partial x_1^2} = -2 < 0$

max

$f_{22} = \frac{\partial^2 y}{\partial x_2^2} = -2 < 0$

The envelope theorem . An app. of the ~~env. theor~~. implicit fn. th.

- how the optimal value for a fn. changes when a parameter of the fn. changes.

ex: Let $y = -x^2 + ax$

we can calculate x^*.

$\frac{dy}{dx} = -2x + a = 0$

$$\frac{a}{2} = \frac{2x}{2}$$

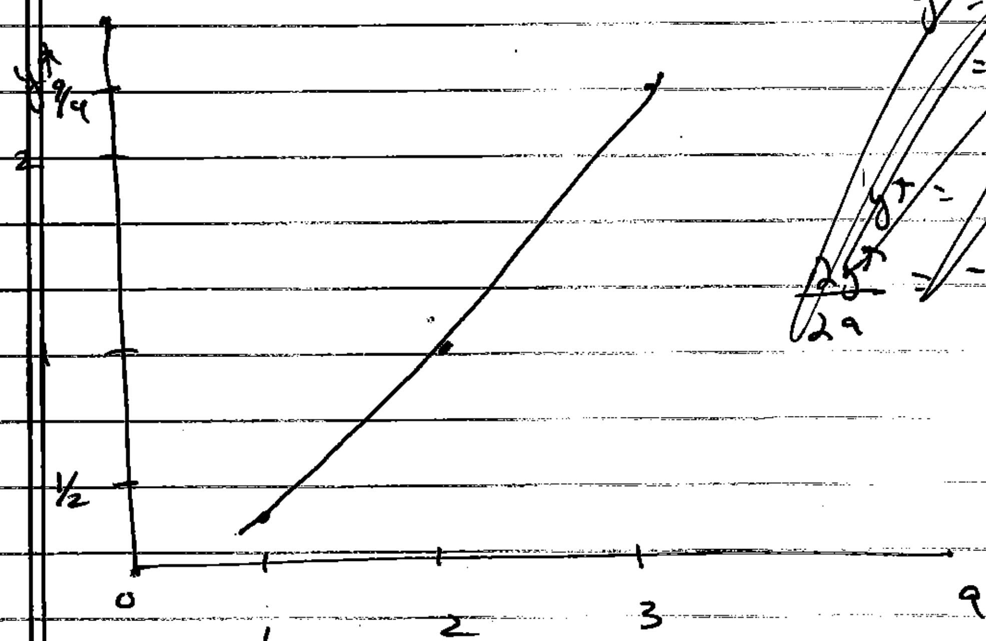
$$\boxed{x^* = \frac{1}{2}a}$$

Now, notice that x^* is a fn. of a. So, notice:

$$\boxed{\begin{array}{l} x^* = \frac{1}{2}a \\ y = -x^2 + ax \end{array}}$$

a	x^*	y^*
0	0	0
1	$\frac{1}{2}$	$\frac{1}{4}$
2	1	1
3	1.5	9/4

$$\cancel{y^* = -\left(\frac{1}{2a}\right)^2 + a\left(\frac{1}{2a}\right)}$$
$$\cancel{= -\frac{1}{4}a^2 + \frac{1}{2}}$$
$$\cancel{y^* = \frac{1}{2} - \frac{1}{4}a^2}$$
$$\cancel{\frac{\partial y^*}{\partial a} = -\frac{1}{2}a}$$



The envelope theorem states that the slope of the rel. b/w y^* and a can be found by calculating the slope of the auxillary rel. found by subst. the respective optimal values for X into the obj fn. & calculating $\frac{\partial y}{\partial a}$

$$y^* = -\left(\frac{1}{2}a\right)^2 + a\left(\frac{1}{2}a\right)$$
$$= -\frac{1}{4}a^2 + \frac{1}{2}$$
$$\boxed{y^* = \frac{1}{2} - \frac{1}{4}a^2}$$

$$\boxed{\frac{\partial y^*}{\partial a} = -\frac{1}{2}a = x^*}$$

If we are interested in how y^* changes as a changes, then calculate the slope of y directly holding X constant at its optimal value, then calculate $\frac{\partial y}{\partial a}$

env. th.

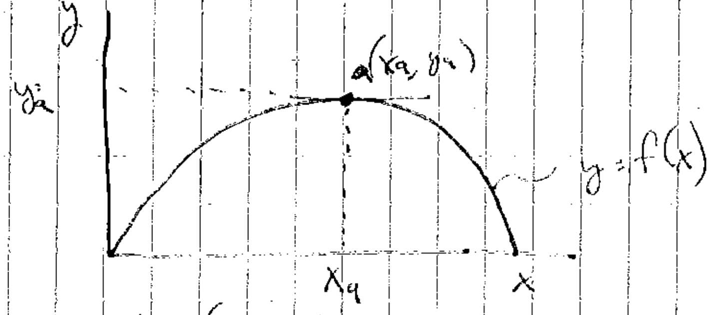
Lecture 1 outline ①

1. What are ~~eco~~ models? (Silberberg, NS) Distinctions
   - truth, usefulness, what is a "good theory"?, testability, math, Popper, assumptions
   - Positive to Normative
   - What tools will we use?
     * ~~Differential~~ Calculus: mostly derivatives, to integration.
     * Game Theory
     * "optimization"
2. What should you learn outside of my lectures.
   (a) NS chapter 2
   (b) For intuition, any intermediate text, like Varian or Nicholson.
3. I will miss several days, so we will have to make them up. I will need to get schedules. The sooner the better.

Maximization ~~Example~~

- Finding the max value for a function w/ several variables ~~requires~~
- Simple case: one variable.

$y = f(x)$

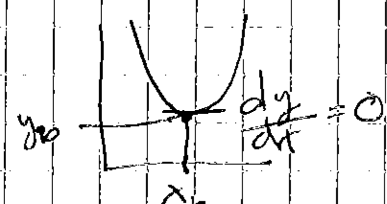


- Visually, point a (x_a, y_a) is the maximum of the function, $f(x)$, b/c y_a is the highest ~~value~~ real number in the set of real numbers contained in $f(x)$.
- ~~The point slope~~ The maximum can also be understood in terms of ~~the~~ the function's slope. The slope is the rate of change in y when x increases by a small amount. When that rate of change is zero, y is no longer rising. A necessary condition for (x_a, y_a) to be the max by the measure of slope is that the slope be 0.

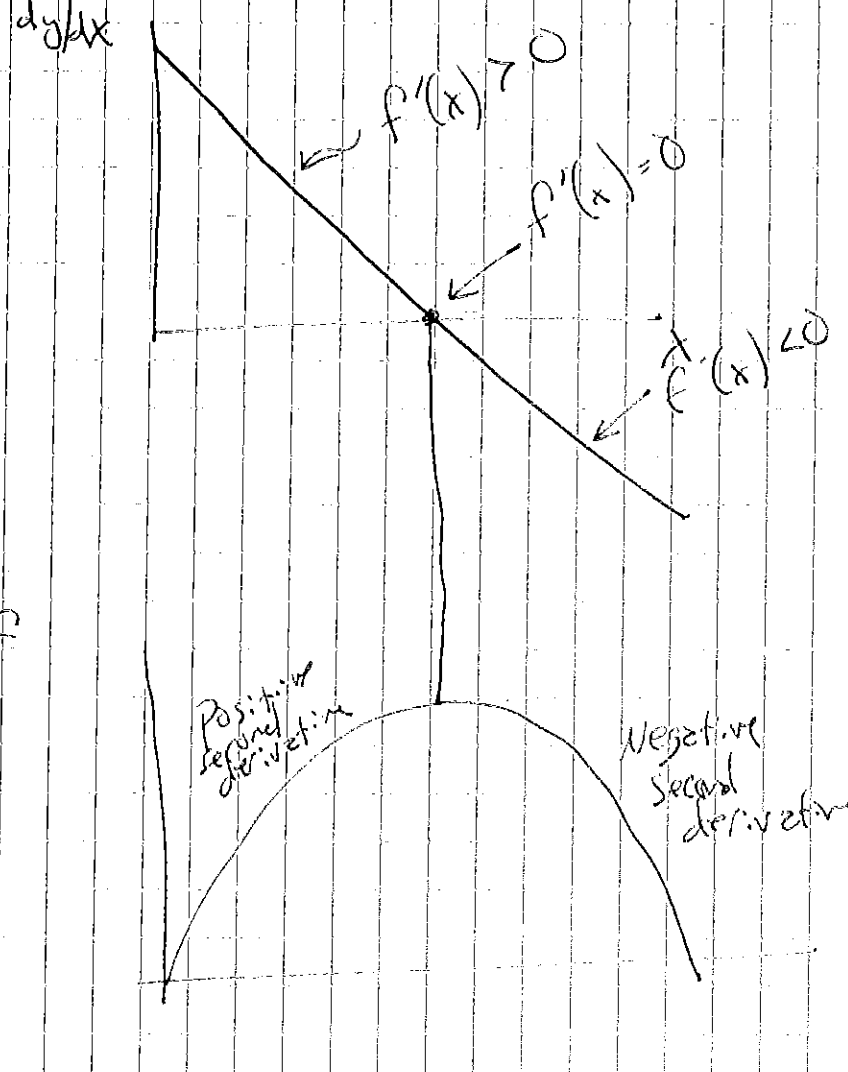
That is, $\frac{dy}{dx} = f'(x) = 0$

$\frac{rise}{run}$

Where the "d" is the derivative of y wrt a change in x. But it is only necessary condition; it isn't sufficient, b/c this also has a slope of 0:



~~What we~~
We can, though, take the derivative function, $f'(x)$, and plot its values.



To determine whether the point you found is a maximum of the function, you need to take the derivative of the slope formula. If it is negative at your solution, then it's a max.

$$\frac{d^2y}{dx^2} < 0$$

Rules for finding derivatives

1. If b is a constant, then:

$$\frac{db}{dx} = 0$$

2. If ~~be~~ is a constant, then

$$\frac{d[b f(x)]}{dx} = b f'(x)$$

3. If b is a constant, then

$$\frac{dx^b}{dx} = b dx^{b-1}$$

4. $$\frac{d \ln x}{dx} = \frac{1}{x}$$

5. $\frac{da^x}{dx} = a^x \ln a$ for any constant a.

6. Assume $f(x)$ & $g(x)$ are two functions & the deriv. exist. Then

$$\frac{d[f(x) + g(x)]}{dx} = f'(x) + g'(x)$$

7. $$\frac{d[f(x) \cdot g(x)]}{dx} = f'(x) g(x) + g'(x) f(x)$$

8. $$\frac{d[f(x)/g(x)]}{dx} = \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}$$

$\frac{f(x)}{g(x)}$

Quotient rule:

$$\boxed{\frac{T'B - B'T}{B^2}}$$

Chain Rule

9. Let $y = f(x)$, & $x = g(z)$, & both deriv. exist. Then:

$$\frac{dy}{dz} = \frac{d[f(g[z])]}{dz} = \frac{df}{dx} \frac{dx}{dz} = \frac{df}{dx} \frac{dg}{dz}$$

Over

Multiple variables

Suppose a person wants to achieve some maximum level of health by consuming combination of protein, carbs & fats. Let y be the measure of health.

$$y = f(C, P, F)$$

The partial derivative is used here as we want to know what happens to health when only one changes.

$$\frac{\partial y}{\partial C} = f_c(C, P, F)$$

the change itself would be:

$$dy = \frac{\partial f}{\partial C} dC$$

the change in your overall health would be equal to the change in your cons. of carb. multiplied by the slope measured in the C direction.

example:

1. $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2 \quad \text{(holding } x_2 \text{ constant)}$$

$$\frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2 \quad \text{(holding } x_1 \text{ constant)}$$

2. $y = f(x_1, x_2) = e^{ax_1 + bx_2}$

$$\frac{\partial y}{\partial x_1} = f_1 = a e^{ax_1 + bx_2}$$

3. $y = f(x_1, x_2) = a \ln x_1 + b \ln x_2$

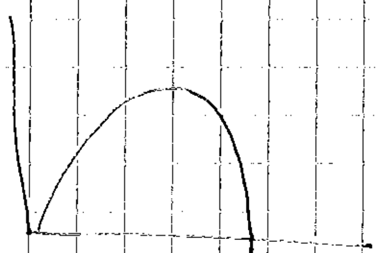
$$\frac{\partial f}{\partial x_1} = f_1 = \frac{a}{x_1}$$

$$\frac{\partial f}{\partial x_2} = \frac{b}{x_2}$$

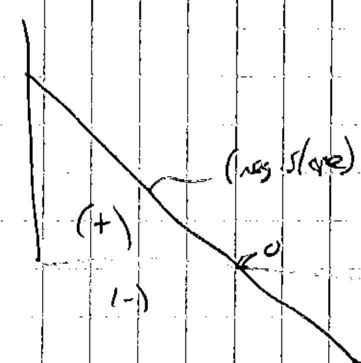
~~Young's Theorem~~

Second Order Partial Deriv.

- Thinking back to univariate case, we are calculating the slope of the slope function.



→



Multivariate: (slope f_i)

$$\frac{\partial(\partial f/\partial x_i)}{\partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2} = f_{ii}$$

1. $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$

$f_1 = 2ax_1 + bx_2$

$f_2 = 2cx_2 + bx_1$

f_{11} $\quad \frac{\partial^2 f}{\partial x_1^2} = f_{11} = 2a + \cancel{bx_1}$

f_{12} $\quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = f_{12} = b$

f_{21} $\quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = \cancel{b}\ f_{21} = b$

f_{22} $\quad \frac{\partial^2 f}{\partial x_2^2} = 2c$

You can do the rest. Notice:

$$f_{12} = f_{21} = 6.$$

Young's Theorem

- The order in which partial differentiation is conducted to evaluate second-order partial derivatives does not matter:

$$f_{ij} = f_{ji}$$

for any pair of variables, x_i & x_j.

- It will also imply that the matrix of all second-order partial derivatives of a function is symmetric

Total Differential

Let $y = f(x_1, \ldots, x_n)$. If all the x's are allowed to vary by a small amount, then the total effect will be the sum of the changes.

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$\boxed{dy = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n}$$

~~analogous to the univariate story~~

more general form of health functions earlier.

$\underline{FOC}$: $f_1 = f_2 = \cdots = f_n = 0$.

This is called the critical point. But could be a min or max. All hilltops are flat, but not all flat places are hilltops.
We will need to look at second partials.

– "For a specific point to provide a (local) maximum value to the function f it must be the case that no small movement in any direction can increase its value."

Implicit Function

We can rewrite a dependent vbl. like this:

$$y = mx + b$$

$$\downarrow$$

$$y - mx - b = 0$$

or

$$f(x, y, m, b) = 0$$

Functions written like this are called implicit fn. b/c the rel. b/w the variables & parameters are implicitly present in the equation rather than calculated.

ex:
$$x + 2y - 4 = 0$$
$$x = 4 - 2y = -2y + 4$$
or
$$\cancel{y = \frac{x-4}{2} = \frac{x}{2} - 2}$$
$$\frac{-2y}{-2} = \frac{x - 4}{-2}$$
$$y = 2 - \frac{x}{2}$$

Derivatives:

Say $f(x, y) = 0$ w/ TD of $f_x dx + f_y dy = 0$

then, $f_y dy = -f_x dx$

$$\boxed{-\frac{dy}{dx} = \frac{f_x}{f_y}}$$

envelope theorem

$a=1$ $y = x - x^2$
$y = x(1-x)$
$\frac{\partial y}{\partial x} = 1 - 2x = 0$
$\frac{1}{2} = \frac{2x}{2} = x^* = 1/2$
So $y^* = 1/2(1-1/2) = 1/4$

An app. of implicit fn theorem. First an ex:

$$y = -x^2 + ax$$

The envelope theorem deals w/ solution values ~~of~~ for y to x when the parameters themselves change.

It is studying how the optimal value of a function changes when the parameter changes. The shortcut is:

$$\frac{dy}{da} = x$$

at x^*, we have:

$$\frac{dy^*}{da} = x^* = 9/2$$

$a = 2$:
$y = -x^2 + 2x$
$\frac{\partial y}{\partial x} = -2x + 2 = 0$
$\frac{\cancel{2}}{\cancel{2}} = \frac{\cancel{2}x}{\cancel{2}}$
$\boxed{x = 1}$

~~$\frac{dy^*}{da} = \frac{\partial y}{\partial a}\{x = x^*(a)\}$~~

Multivariate

$$y = f(x_1, \dots, x_n; a)$$

Finding optimal value for y consists of solving n FOC eq. of the form:

$$\frac{\partial y}{\partial x_i} = 0 \quad (i = 1, \dots, n)$$

to yield $(x_1^*, x_2^*, \dots, x_n^*)$ that depend implicitly on a. Subst. all X's into obj. fn. to get y^*:

$$y^* = f(x_1^*, x_2^*, \dots, x_n^*)$$
$$f(x_1^*(a), x_2^*(a), \dots, x_n^*(a); a)$$

TD w.r.t. a:

$$\frac{dy^*}{da} = \frac{\partial f}{\partial x_1^*}\frac{dx_1^*}{da} + \frac{\partial f}{\partial x_2}\frac{dx_2^*}{da} + \dots + \frac{\partial f}{\partial x_n^*}\frac{dx_n}{da}$$

But, b/c of FOC (i.e. $\frac{\partial f}{\partial x_i} = 0 \ \forall i$) all the terms are 0 at their optimal values, giving us the envelope result:

$$\frac{dy^*}{da} = \frac{\partial f}{\partial a}$$

where the derivative is evaluated at optimal values for X.

Constrained Max

Let an agent have obj fn:

$$y = f(x_1, x_2, \ldots, x_n)$$

~~so~~ facing constraint that only certain values of be used:

$$g(x_1, \ldots, x_n) = 0$$

Choose values of $x_1, \ldots, x_n$ that maximizes y.

Lagrangian method

$$\mathcal{L} = f(x_1, \ldots, x_n) + \lambda\, g(x_1, \ldots, x_n)$$

the Lagrangian multiplier. Notice that when the constraint holds $\mathcal{L}$ and f are equal b/c $g(x, \ldots, x_n) = 0$.

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1 + \lambda g_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = f_2 + \lambda g_2 = 0$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial x_n} = f_n + \lambda g_n = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1, \ldots, x_n) = 0$$

~~Math Review~~ (11)

~~Maximization~~

These FOC equations are the conditions for a critical point for the $\mathcal{L}$ function. Notice there are $n+1$ equations (one for each x and λ) & $n+1$ unknown variables (x & λ). The equations allow us to solve for $x, \dots, \lambda$. Solution will have two properties:

1. x's will obey constraints b/c the last equation ~~xxx~~ imposes that condition.
2. among those x' values satisfying that constraint, those that also solve the equations will make $\mathcal{L}$ as large as possible.

Interpretation of λ. Rewrite the FOC equations:

$$\frac{f_1}{-g_1} = \frac{f_2}{-g_2} = \dots = \frac{f_n}{-g_n} = \lambda$$

→ ~~λ is equal to the ratio of f_i to g_i~~

At the critical point, the ratio of f_i to g_i is the same for every x_i

Numerator: Marginal contribution of each x_i to the function f.
"Marginal benefit" of one more unit of x to f.

~~denominator~~

The denominator will make more sense when we see it in ~~xxxx~~ actual economic applications. But in our applications, it will usually be a marginal cost.

ex: assume the constraint requires that total spending on x_1 and x_2 to be given by a fixed dollar amount, F. Hence:

$$p_1x_1 + p_2x_2 = F$$

where p_i is the market price of good x_i. The constraint would be rewritten as:

$$g(x_1, x_2) = F - p_1x_1 - p_2x_2 = 0$$

Therefore, $\frac{\partial g}{\partial x_1} = -p_1$

And so,

$$\frac{f_1}{g_1} = \lambda = \frac{f_1}{p_1} = \lambda$$

Here we see that the derivative, g_1, reflects the per unit cost of an additional unit of x_1

Lagrange multiplier is, now, a MB/MC ratio

$$\lambda = \frac{MB \text{ of } x_i}{MC \text{ of } x_i}$$

Duality

Key insights in both the theory of the consumer & the theory of the firm are borne from seeing the rel. b/w obj. fn. & constraints can be reversed w/out changing the solution — called duality theory. As we will depend heavily on it, we should try to understand it abstractly first.

(13)

Any ~~good~~ constrained maximization problem has an associated dual problem in constrained minimization.

An example we will encounter:

- the consumer will maximize utility subject to his BC.

- it can be rephrased as an expenditure min. subject to a "utility" constraint where utility is held at the value solved in the primal problem.

other apps: a firm will max output subject to a cost constraint, or it will min. cost subject to an output constraint. Two perspectives on the same decision, but through duality "theory" so to speak we will learn more about behavior embedded in our models.

ex: let $y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$

$x_1 + x_2 = 1$

$\mathcal{L} = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5 + \lambda(1 - x_1 - x_2)$

foc: $\dfrac{\partial \mathcal{L}}{\partial x_1} = -2x_1 + 2 - \lambda = 0$

$\dfrac{\partial \mathcal{L}}{\partial x_2} = -2x_2 + 4 - \lambda = 0$

$\dfrac{\partial \mathcal{L}}{\partial \lambda} = 1 - x_1 - x_2 = 0$

Solve for $x_1^*, x_2^*, \& \lambda^*$

$\dfrac{-2x_1 + 2}{-2} = \dfrac{-2x_2 + 4}{-2}$

$x_1 - 1 = x_2 - 2$

$\boxed{x_1 = x_2 - 1}$

$1 - x_1 - x_2 = 0$

$1 - [x_2^* - 1] - x_2^* = 0$

$\dfrac{2}{2} = \dfrac{2x_2^*}{2}$

$\boxed{x_2^* = 1}$

$1 - x_1^* - x_2^* = 0$

$1 - 1 = x_1^*$

$\boxed{x_1^* = 0}$

$\lambda^* = 2 - 2x_1$

$= 2 - 0$

$\boxed{\lambda^* = 2}$

If $x_1 = 0$, & $x_2 = 1$, then $y = -(0)^2 + 2(0) - (1)^2 + 4(1) + 5 = 8$

$\boxed{8}$

~~min x_1+x_2 s.t. $y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$~~

Envelope Th. in Constrained Max.

Suppose we seek max value of:

$$y = f(x_1, \dots, x_n; a)$$

$$\text{s.t.} \quad g(x_1, \dots, x_n; a) = 0$$

$$\mathcal{L} = f(x_1, \dots, x_n; a) + \lambda g(x_1, \dots, x_n; a)$$

and solve for FOC for optimal, constrained, values of X_i^*.

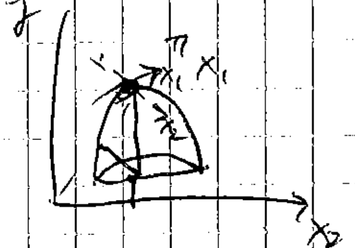
Envelope th: Also, when we solve for X_i^* & put back into $\mathcal{L}^*$, we get y^*.

Now, say we want to know how the optimal value of y^* changes if we vary the a parameter. (ex, how will a consumer's utility [sitting here?] change if we vary the price of good X?)

$$\boxed{\frac{\partial y^*}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}(x_*^1, \dots, x_*^n)}$$

This is the env. theorem applied.

2 vbls



$f_{11} < 0$
$f_{22} < 0$

are the intuitive counterparts. But what about: f_{12} or f_{21} ?

$\boxed{\text{cross partials}}$

formal analysis

~~$dy = f_1 dx_1 + f_2 dx_2$~~

① T.D.

② T.D.

let $y = f(x_1, x_2)$

$dy = f_1 dx_1 + f_2 dx_2$

$d^2y = f_{11} dx_1^2 + f_{12} dx_1 dx_2 + f_{21} dx_2 dx_1 + f_{22} dx_2^2$

Young's Thm:

$f_{12} = f_{21}$

$$\boxed{d^2y = f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + f_{22} dx_2^2}$$

we need $d^2y < 0$ for SO. ∴ sum of RHS must be neg. What conditions?

(i) let $dx_2 = 0$

$\underset{(-)}{f_{11}} \underset{(+)}{dx_1^2}$

and by symmetry:

① $f_{11} < 0$

② $f_{22} < 0$

③ interior terms: $2 f_{12} dx_1 dx_2$. What?

Concavity in unconstrained max:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

4. Neg/Pos. Def. We say the matrix is neg/pos def or semi-definite.

<u>Pos def</u>: if all leading p.m. are pos.
<u>Neg def</u>: " " " " are alternating in sign starting w/ minus.

So, back to unconstrained max. We need to know if our function is at a max. This is in matrix algebra equiv to whether the det. of H is neg def.

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

<u>1st</u> Princ. leading minor: $f_{11} < 0$ by assumption.
<u>2nd</u> " " " : $f_{11}f_{22} - f_{21}f_{12}$

$f_{12} = f_{21}$

$$\boxed{f_{11}f_{22} - f_{12}^2 > 0}$$

this condition is, w/ $f_{11} < 0$ & $f_{22} < 0$, enough to ensure that $d^2y < 0$ at the critical points, x_1^* & x_2^*

SOC:

Let $y = f(x)$

$\frac{dy}{dx} = f'(x) = 0$ ← necessary.

$dy = f'(x)dx$

to det. if it is a max & not a min, then

$\frac{d(dy)}{dx} = \cancel{f'(x)dx}\ \frac{d[f'(x)\,dx]}{dx} = f''(x)dx$

$\frac{d^2y}{dx} = f''(x)dx$

$\boxed{d^2y = f''(x)dx^2}$

* $\boxed{d^2y < 0}$ implies that $f''(x)\,dx^2 < 0$, and $dx^2 > 0$, so SOC is $f''(x) < 0$.
└ required.

Soo

2 vbls: $y = f(x_1, x_2)$

$\frac{\partial y}{\partial x_1} = f_1 = 0$

$\frac{\partial y}{\partial x_2} = f_2 = 0$

(1) Total diff y wrt all arguments
$dy = f_1 dx_1 + f_2 dx_2$

(2) 2nd Differential of

$d^2y = f_{11}dx_1^2 + f_{12}dx_1dx_2 + f_{21}dx_1dx_2 + f_{22}dx_2^2$

(3) Young's Th.
$f_{12} = f_{21}$

(4) $d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$

Must be negative (i.e. $d^2y<0$). Therefore, what conditions yield that?

$$f_{11}<0$$
$$f_{22}<0$$

f_{12}? the cross partial effects.

$$|D| = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \underbrace{f_{11}f_{22}}_{(+)} - f_{12}^2 > 0$$

the neg. condition is: $\boxed{f_{11}f_{22} - f_{12}^2 > 0}$

Constraint Max:

$\max_{(x_1, x_2)} y = f(x_1, x_2)$ st. $b_1 x_1 + b_2 x_2 = c$

$$\mathcal{L} = f(x_1, x_2) + \lambda(c - b_1 x_1 - b_2 x_2)$$

foc: $\frac{\partial \mathcal{L}}{\partial x_1} = f_1 - \lambda b_1 = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = f_2 - \lambda b_2 = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = c - b_1 x_1 - b_2 x_2 = 0$

then we solve the system to find x_1^*, x_2^* & λ^*.
- 3 unknowns in 3 equations.

SOC: Now we need to know if at those solutions, any movements away will cause values to fall, using 2nd ~~partials~~ total diff.

$$dy = f_1 dx_1 + f_2 dx_2$$

$$\boxed{d^2 y = f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + f_{22} dx_2^2}$$

BUT, w/ constraint $b_1 x_1 + b_2 x_2 = c$, there are explicit constraints which x_1/x_2 can take on, & the constraint forces t/radeoffs to be made if you move to increase one: for instance, at the point where the constraint is binding, $x_1 \uparrow$ will only occur if x_2 falls by:

Tricks
- Two substitutions we'll make.

① $\longrightarrow$

$b_1 x_1 + b_2 x_2 = c$

$\frac{b_1 x_1}{b_1} = \frac{c - b_2 x_2}{b_1}$

$$\boxed{x_1 = \frac{c}{b_1} - \frac{b_2}{b_1} x_2}$$

So, dx_1 cannot be anything.

Therefore, we must substitute constraints into d^2y. But x_1 & x_2 aren't in there, only dx_1 & dx_2 are. Therefore,

$$\frac{dx_1}{dx_2} = -\frac{b_2}{b_1}$$

$$dx_1 = \frac{-b_2}{b_1} dx_2$$

or

$$\boxed{dx_2 = \frac{-b_1}{b_2} dx_1}$$

AND

Second subst from the foc

foc:

$f_1 - \lambda b_1 = 0$

$f_2 - \lambda b_2 = 0$

$$\lambda = \frac{f_1}{b_1} = \frac{f_2}{b_2}$$

$$\boxed{\frac{f_1}{f_2} = \frac{b_1}{b_2}}$$

Now, subst the constraint into our SOC to get

$$d^2y = f_{11}dx_1^2 + 2f_{12}\,dx_1dx_2 + f_{22}dx_2^2$$

$$= f_{11}dx_1^2 + 2f_{12}dx_1\left[-\frac{b_1}{b_2}dx_1\right] + f_{22}\left[\frac{-b_1}{b_2}dx_1\right]^2$$

$$= f_{11}dx_1^2 - \frac{2b_1}{b_2}f_{12}dx_1^2 + f_{22}\frac{b_1^2}{b_2^2}dx_1^2$$

subst. foc

$$= f_{11}dx^2 - 2\frac{f_1}{f_2}f_{12}dx_1^2 + f_{22}\left(\frac{f_1}{f_2}\right)^2 dx_1^2$$

For $\downarrow d^2y$ $(-)$

$$= \underbrace{\frac{dx_1^2}{f_2^2}}_{(+)}\underbrace{\left(f_{11}f_2^2 - 2f_1f_2f_{12} + f_1^2f_{22}\right)}_{(-)}$$

For $d^2y < 0$, it must be that

$$f_{11}f_2^2 - 2f_1f_2f_{12} + f_1^2f_{22} < 0$$

What gives us this? Quasi restriction classify to a class of functions called "QC".

QC = quasi the set of all points for which such a function takes on a value greater than any specific constant is a convex set.

Show picture

We use a modified Hessian called the bordered Hessian.

Let $f(x_1, \ldots, x_n)$

s.t.

$g(x_1, \ldots, x_n) = 0$

foc: $f_i + \lambda g_i = 0$

SOC:

$$H_b = \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ g_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ g_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

For a max, SOC require the H_b ~~det~~ to be neg. def.

~~Hbt~~ 1. Princ. minor $= < 0$
2. " " $= > 0$

For a min, all princ. minors except 1st should be neg.

To show this is neg., load 2nd partials into matrix. ~~Concavity neg~~

Hessian matrix

1. Symmetric
2. Formed by all 2nd order partials of a fn.

If f is continuous to twice diff of n vbls, its Hessian is

$$H(f) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

Who need to know if our 2-vbl fn, f, is concave at the critical point. That in matrix language is the same as wanting to know if a matrix is pos. def. So, briefly, some matrix algebra

Determinant of a square matrix is

1. notation: If A is matrix, $|A|$ is its det.
2. $|A|$ is some "scalar," or what we typically call in normal words "number" like 4 or 3.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad |A| = (a_{11}a_{22} - a_{12}a_{21})$$

or

$$a_{11}a_{22} - a_{12}^2$$

3. leading pric. minors of $n \times n$ square matrix A

are the series of det. of p rows & p colums, where $p = 1, n$. If A is 2×2, then:

1st princ lead minor $= a_{11}$

2nd " " " $= a_{11}a_{22} - a_{12}^2$

Lecture 3

1. Constrained Max
   - Second order Conditions : Bordered Hessian & QC. Show pictures of 2.10 (p. 52)
   - example of unconstrained & constrained.
2. Homogeneous Fn.
   - definition of the "H^k"; k=1 and k=0 cases
   - H_0 derivative properties ex.
   - Euler's Theorem
   - Homothetic fn.
3. Preferences & Utility
   - Axioms of RATC:
     * Completeness
     * Transitivity
     * Contiguity
   - Utility or "Preference Function" (Tullock)
     * ordinal vs. cardinal
     * nonuniqueness & monotonic transformation
     * consumption & ceteris paribus
     * Arguments of utility fn.
     * Definition.
     * Visualize it w/ economic goods on 2-dimen.
   - Trades & Substitution
     * Indifference Curves & the MRS (define)
     * the map ("level surface")
     * transitivity & non-transitivity on the map
     * Convexity of the map.
     * Balance & convexity.
   - Mathematical Derivation
     * MRS from total diff of utility fn.
     * Diminishing MRS - intuition & formal
     * examples (1)
   - Diff. Preferences are represented by diff. Utility fn.
     * CD
     * Perfect Subst.
     * Perfect Complement
     * CES utility

Homogeneous fn.

$f(x_1, \dots, x_n)$

A fn. is said to be Homog. of degree K if:

$$f(tx_1, tx_2, \dots, tx_n) = t^K f(x_1, \dots, x_n)$$

- when K=1, then doubling its arguments by t doubles the value of the fn. itself
- when K=0, it'll have no effect.

H. of Deriv.

If a fn. is h. of degree K & has a deriv, the partials of the fn. will be homog. of degree K-1.

take: $f(tx_1, \dots, tx_n)$

$$\frac{\partial f(tx_1, \dots, tx_n)}{\partial x_1} \cdot t = t^K \frac{\partial f(x_1, \dots, x_n)}{\partial x_1}$$

or

$$f_1(tx_1, \dots, tx_n) = t^{K-1} f_1(x_1, \dots, x_n)$$

if K=1, then it's also

$$f_1(tx_1, \dots, tx_n) = f_1(x_1, \dots, x_n)$$

partial derivatives are the same

Euler's Theorem

Another useful feature of homogeneous functions can be shown by differentiating the definition for homogeneity wrt proportionality factor, t. ~~Diff~~ Take this equation from earlier:

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n)$$

Diff RHS wrt t first:

$$\frac{\partial (RHS)}{\partial t} = k t^{k-1} f(x_1, x_2, \dots, x_n)$$

Now the LHS:

$$\cancel{\frac{\partial f(\cdot)}{\partial t} = t f_1(tx_1, \dots, tx_n) + \dots + t f_n(tx_1, \dots, tx_n)}$$

$$\frac{\partial (LHS)}{\partial t} = x_1 f_1(tx_1, \dots, tx_n) + \dots + x_n f_n(tx_1, \dots, tx_n)$$

Set equal, & let $t=1$:

$$\boxed{k f(x_1, \dots, x_n) = x_1 f_1(x_1, \dots, x_n) + \dots + x_n f_n(x_1, \dots, x_n)}$$

("Euler's Theorem for homogeneous fn."

- What's it mean? For a homogeneous fn, there is a definite rel. b/w the values of the function and the values of its partial derivatives.
- Several important economic rel. among fn. are based on this obs.
- "Discovering behavior w/ math" – Kind of cool.

Homothetic functions

defn: monotonic transformation preserve the order of the rel. b/w the arguments of a fn. & the value of that fn.

So, if a certain set of x's yield larger values of y, then a monotonic transf. will preserve that order of y.

ex: $f(x, y) = xy$

if we multiply (xy) by 2, the xy combos will still cause $f(x,y)$ to move up or down in the same order as before.

ex: $x=1$, $y=1$, $f(x,y)=1$
$2(xy) = 2*1 = f(x,y) = 2$

$x=2$, $y=1$, $f(x,y) = 2$ $\quad 2>1$
$2(xy) = 2\cdot(2\cdot 1) = 4$ $\quad 4>2$

order preserved.

or $f(x,y) = xy$
$(xy)^2$

But not $-1(xy)$.

Now higher xy give smaller values of $f(x,y)$.

take implicit fn $f(x,y)=0$ & total diff

$$f_x dx + f_y dy = 0$$
$$f_x dx = -f_y dy$$
$$\cancel{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

If f is H^k, its partials are H^{k-1}

$$\frac{dy}{dx} = -\frac{t^{k-1} f_x(tx, ty)}{t^{k-1} f_y(tx, ty)} = -\frac{f_x}{f_y}\frac{(tx, ty)}{(tx, ty)}$$

Now let $t = 1/y$

$$\frac{dy}{dx} = -\frac{f_x(\frac{x}{y}, 1)}{f_y(\frac{x}{y}, 1)}$$

- Which shows that the tradeoff depends only on the ratio of x/y, not on their abs. values.
- If we apply any monotonic transformation (F w/ $F'>0$) to the original Homogeneous fn, we have:

$$\frac{dy}{dx} = -\frac{F' f_x(x/y, 1)}{F' f_y(x/y, 1)} = -\frac{f_x(\frac{x}{y}, 1)}{f_y(\frac{x}{y}, 1)}$$

- Which means the monotonic transformation does not change the fact that $\frac{dy}{dx}$ is a fn. of just the ratios of x to y.

Axiom 3

Continuity — If A < B, ~~and~~ then "situations close to A" are < B.

More of a mathematical assumption. It rules out knife-edge preferences that make it difficult to build a math theory of choice.

Basically, if A < B, then A + ε > B too. Helps us w/ marginalism.

Utility — it is a function ~~that~~ showing a person ranking her preference over all situations from the least to the worst desirable.

* Mention Sister Theresa, Bertha, & Christian writer Piet.

Utility is not happiness, ~~[illegible]~~ or not exactly. It is preferences.

a. Nonuniqueness. Utility simply RANKS the order with numbers, which are the ~~order~~ value of the function.

$U(A) = 2$ $U(A) = 1$

$U(B) = 5$ $U(B) = 700$ million

Same utility fn.

Utility is unique only up to a monotonic transformation. So you could take a boring utility fn like this, square it, & the analysis we do won't change.

$U(x,y) = f(x,y) = \sqrt{x+y}$ "Same utility fn"

$U(x,y) = [f(x,y)]^2 = x + y$

Assign HW#1

2.1, 2.2, 2.3, 2.4

Due Tuesday next week.

Preferences & Utility.

Consumer theory is built on notion of utility, which is itself based on the idea of "preferences." We make 3 assumptions about people's preferences now.* We say rational behavior is based on these assumptions about personal preferences.

Axiom 1: Completeness: If A & B are any two situations, the individual can always specify exactly one of these poss:

i. $A \succ B$
ii. $B \succ A$
iii. A & B are equally attractive.

so: no indecision. They can completely make up their minds about the desirability of 2 alt. tell story of kids' indecision on which movie to watch. May not always be true. It is, though, an assumption.

Axiom 2: Transitivity: If $A \succ B$, and $B \succ C$, then $A \succ C$.

"internally consistent preferences."

money pump: If you prefer 10¢ to 5¢ and
5¢ to 1¢ and
1¢ to 10¢

that would violate transitivity. That means if

Lecture 2: Math

1. ~~Constrain~~ Constrained max
   a. Lagrangian
   b. interpretation of λ
2. Duality
   a. primal vs. dual prob.

ex: let $y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$

constraint:

$x_1 + x_2 = 1$

primal

$\max_{(x_1,x_2)} \mathcal{L} = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5 + \lambda(1 - x_1 - x_2)$

FOC $\frac{\partial \mathcal{L}}{\partial x_1} = -2x_1 + 2 - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = -2x_2 + 4 - \lambda = 0$

$\Rightarrow \lambda = \frac{2-2x_1}{2} = \frac{4-2x_2}{2}$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x_1 - x_2 = 0$

$\boxed{x_1 = 1 - 2 + x_2}$

$1 - [1 - 2 + x_2] - x_2 = 0$

$1 - x_1 = 2 - x_2$

$1 - 2 + x_2 = x_1$

$x_2 - 1 = x_1$

$\boxed{x_1 = x_2 - 1}$

$1 - [x_2 - 1] - x_2 = 0$

$1 - x_2 - x_2 + 1 = 0$

$\frac{2}{2} = \frac{2x_2}{2}$

$\boxed{x_2^* = 1}$

$1 - x_1^* - 1 = 0$

$\boxed{x_1^* = 0}$

$\lambda = 2 - 2x_1^*$

$= 2 - 2(0)$

$\boxed{\lambda^* = 2}$

$y^* = -(0)^2 + 2(0) - (1)^2 + 4(1) + 5$

$y^* = 0 + 0 - 1 + 4 + 5$

$\boxed{y^* = 8}$

dual

~~min~~ let $Z = x_1 + x_2$ + constraint is: $-x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5 = 8$

$\mathcal{L} = x_1 + x_2 + \lambda(8 + x_1^2 - 2x_1 + x_2^2 - 4x_2 - 5)$

FOC: $\frac{\partial \mathcal{L}}{\partial x_1} = 1 + \lambda(2x_1 - 2) = 0 \rightarrow \frac{\lambda(2x_1 - 2)}{(2x_1 - 2)} = \frac{-1}{2x_1 - 2}$

$\boxed{\lambda^* = -\tfrac{1}{2}x_1 + \tfrac{1}{2}}$

$\boxed{\lambda^* = \tfrac{1}{2} - \tfrac{1}{2}x_1}$

$\frac{\partial \mathcal{L}}{\partial x_2} = 1 + \lambda(2x_2 - 4) = 0$

$\lambda^* = \frac{-1}{2x_2 - 4} = \frac{1}{2}x_2 + \frac{1}{4}$

$\lambda^* = \frac{1}{4} - \frac{1}{2}x_2$

Second order conditions w/ unconstrained mv. case

Let $y = f(x)$

$$\frac{dy}{dx} = f'(x)$$

$$\boxed{dy = f'(x)dx}$$

$$\frac{d(dy)}{dx} = f''(x)dx$$

$$\boxed{d^2y = f''(x)dx^2}$$

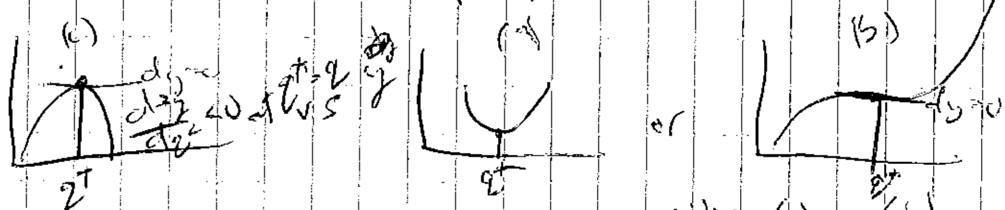
For it to be a max, $\boxed{d^2y<0}$, b/c it must be that at critical point, changing moment causes slope of $\frac{dy}{dx}$ to fall, not rise. This requires that:

$$\boxed{\underbrace{f''(x)}\underbrace{dx^2}_{(+)} < 0}$$

(1) we know $dx^2 > 0$, ∴

⟶ (2) $f''(x) < 0$.

For one var, assuming $f''(x) < 0$ is sufficient. To have a "concave" shape at the critical point.



$\uparrow$ max

neither (a) or (b) is a max, though $dy = 0$.