Chapter 5: Interest Rates

Fundamental questions:

1) How do we handle non-annual cash flows and interest rates?
2) What factors affect the interest rate on a security?

5.1 Interest Rate Quotes and Adjustments

A. The Effective Annual Rate

Effective interest rate: actual interest rate for period

Ex.

1) Assume effective interest rate for year is 9%. If deposit $100 for a year, have $109 at end of the year
2) Assume effective interest rate for month is 2%. If deposit $100 for a month, have $102 at the end of the month.

1. Adjusting the Discount Rate for Different Time Periods

2. General Equation for Discount Rate Period Conversion

\[ r(t) = (1 + r)^n - 1 \]  

(5.1)

Notes:

1) \( r(t) \) is the effective rate for period “t”. I express “t” in terms of years.

Ex.

\[ r \left( \frac{1}{2} \right) = \text{effective semiannual rate (half a year)} \]
\[ r \left( \frac{1}{12} \right) = \text{effective monthly rate (one-twelfth of a year)} \]

2) \( r \) is effective interest rate that converting to a different period (length of time)
3) $n$ is adjustment factor: time converting to/time starting with

Examples:

- If want a rate that is twice as long as the one you start with, $n = 2$
- If want a rate that is six times as long, $n = 6$
- If want a rate that is one half as long, $n = \frac{1}{2}$
- If want a rate that is one-twelfth as long, $n = \frac{1}{12}$

4) As noted in chapter 4, must use an effective interest rate that matches the frequency of the cash flows for any series of cash flows (annuities and perpetuities).

5) As noted in chapter 4, can use any effective interest rate for taking the present or future value of a single cash flow.

6) Footnote 1 is important. APY (annual percentage yield) is the terminology that banks are required to use for the effective annual interest rate on deposits. This is required by the Truth in Savings Act of 1993.

7) On quizzes and exams, you are sometimes only required to set up problems without solving them. It is important in this case to make it clear which effective interest rate you are using.

B. Annual Percentage Rates (APR)

APR: an interest rate that ignores the effect of compounding

$$r(t) = \frac{APR}{k}$$

(5.2)

Notes:

1) can’t use APRs in time-value-of-money calculations. Only use of an APR is to calculate an effective interest rate
2) $k =$ number of compounding periods per year
3) $t =$ time frame of interest rate in years = $1/k$
4) any time the interest rate is an APR, must start with this equation to convert to an effective interest rate

Khan Academy:

Annual Percentage Rate and Effective APR
Introduction to compound interest and e

C. Examples
1. Assume an APR of 6% per year with semiannual compounding. What is the effective annual interest rate and the effective monthly interest rate on this account?

\[ r(t) = \frac{APR}{k} \]

\[ r(\frac{1}{2}) = \frac{0.06}{2} = 0.03 \]

=> effective semiannual rate (half a year) is 3%

\[ r(t) = (1 + r)^n - 1 \]

\[ r(1) = (1.03)^2 - 1 = 0.0609 \]

\[ r(\frac{1}{2}) = (1.03)^{0.5} - 1 = 0.004939 \]

**Video Solution**

Note: \( r(\frac{1}{2}) = 0.03 \), \( r(1) = 0.0609 \), and \( r(\frac{1}{2}) = 0.004939 \) are equivalent

=> end up with same amount of money at the end

Ex. If invest $100 for a year, then your account balance at the end of the year equals:

\[ V_1 = 100(1.03)^2 = 100(1.0609) = 100(1.004939)^{12} = 106.09 \]

Note: result is same if take future value at semiannual rate for two periods or annual rate for one period or monthly rate for 12 periods.

=> all take value one year into future

**Video Solution**

2. Eight months from today you want to make the first of 14 quarterly withdrawals from a bank account. Your first withdrawal will equal $10,000 and each subsequent withdrawal will grow by 1% each. How much do you need to deposit today if the account pays an APR of 9% with monthly compounding?
Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[
n(t) = \frac{APR}{k}
\]
\[
r(t) = (1 + r)^n - 1
\]
\[
r(t) = (1.0075)^{12} - 1 = 0.022669
\]
\[
V_t = \frac{c}{r-g}\left(1 \left(\frac{1+g}{1+r}\right)^N\right)
\]
\[
V_{5mo} = \frac{10,000}{.022669-.01}\left(1 \left(\frac{1.01}{1.022669}\right)^{14}\right) = 126,401.27
\]

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[
V_t = \frac{c_n}{(1+r)^N}
\]
\[
V_0 = \frac{126,401.27}{(1.0075)^5} = 121,766.03
\]

3. What if you want to make the first withdrawal one month from today (and nothing else changes)?

Q: Will the amount you deposit be larger or smaller if 1st withdrawal is in one month instead of eight months? Larger
Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ r_{1,73}^{(t)} = 0.0075; \quad r_{4}^{(t)} = 0.022669; \quad PV_{2mo} = 126,401.27 \]

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ V_t = C \times (1 + r)^n \]

\[ V_0 = 126,401.27(1.0075)^2 = 128,304.39 \]

**Video Solution**

4. A bond matures for $1000 three years and ten months from today. The annual coupon on the bond equals $60 but coupons are paid semiannually. What is the value of the bond if it earns a return of 8% per year?
Steps: 1) **timeline**, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[
r(t) = (1 + r)^n - 1
\]
\[
r(\frac{1}{2}) = (1.08)^{\frac{1}{2}} - 1 = .03923
\]

**Coupons:**

\[
V_t = \frac{C}{r-g} \left( 1 - \left( \frac{1+g}{1+r} \right)^N \right)
\]
\[
V_{-2\text{mo}} = \frac{30}{.03923} \left( 1 - \left( \frac{1}{1.03923} \right)^8 \right) = 202.6257
\]
\[
V_t = C \times (1 + r)^n
\]
\[
V_0 = 202.6257(1.08)^{2/12} = 205.2415
\]

**Par:**

\[
V_t = \frac{C_n}{(1+r)^n}
\]
\[
V_0 = \frac{1000}{(1.08)^{\frac{33}{12}}} = 744.5187
\]

Price = 205.24 + 744.52 = 949.76

**Video Solution**

**Calculator:**

- **V-2 mo**: 30 = PMT, 1000 = FV, 8 = N, 3.923 = I% => PV = 937.6555
- **V0**: 937.6555 = PV, 8 = I%, 2/12 = N => FV = 949.76

D. Practicing N and n

1. Assume you want to take out a loan and can afford to make monthly payments of $100 each starting five months from today that will continue through three years and three months from today. Assume you want to determine the amount you can borrow. Which equation(s) would you use? What would you use for “N” or “n” in each equation?
Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.
A: 1) present value of annuity, N = 35; 2) present value of single cash flow, n = 4

2. Assume your lender will allow you to pay off your loan (in number 1 above) by making payments of only $10 per month (still starting five months from today and continuing through three years and three months from today). You would then make a balloon payment three and a half years from today to pay off the balance of the loan. Assume you want to determine the size of the balloon payment. Which equation(s) would you use? What would you use for “N” or “n” in each equation?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.
A: 1) future value of annuity ($90 payments), N = 35; 2) future value of a single cash flow, n = 3

5.2 Application: Discount Rates and Loans

A. Computing Loan Payments

An important statement you might overlook: “When the compounding interval for the APR is not stated explicitly, it is equal to the interval between payments.”

B. Computing the Outstanding Loan Balance

Concept Check: all

5.3 The Determinants of Interest Rates

A. Inflation and Real Versus Nominal Rates

\[
r_r = \frac{r - i}{1 + i}
\]

(5.5)

where:

\[
r = \text{nominal interest rate} = \text{rate at which money grows}
\]

\[
i = \text{inflation rate} = \text{rate at which prices change}
\]

\[
r_r = \text{real interest rate} = \text{rate at which purchasing power grows}
\]

Note: can use expected or realized rates

Ex. Assume that the nominal interest rate is 6% per year and that inflation is 5% per year. What is the real interest rate?
$r_r = 0.00952 = \frac{0.06 - 0.05}{1.05}$

Note: the difference between the nominal rate and the inflation rate is a pretty good approximation of the real rate if inflation is low.

Khan Academy:

Real and nominal return

Q: Example in book calculates real interest rate in 2011 was – 2.62%. Why would anyone invest at a negative real interest rate?

B. Investment and Interest Rate Policy

Key issue: interest rates affect net present values of investment opportunities

=>$ since most investment require outflow today and inflows in the future, NPV and interest rates move in opposite directions

1. Monetary Policy, Deflation, and the 2008 Financial Crisis

Key issue: Fed uses inverse relationship between interest rates and net present values to stimulate or cool the economy
C. The Yield Curve and Discount Rates

Key issues:

=> interest rates vary by maturity
=> term structure
=> yield curve

\[ V_0 = \frac{C_n}{(1+r_n)^n} \]  \hspace{1cm} (5.6)

\[ V_0 = C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)^2} + \cdots + \frac{C_N}{(1+r_N)^N} \]  \hspace{1cm} (5.7a)

\[ V_0 = \sum_{n=0}^{N} \frac{C_n}{(1+r_n)^n} \]  \hspace{1cm} (5.7b)

Important Warning: All of our shortcuts for computing present values (annuity and perpetuity formulas) are based on discounting all cash flows \textit{at the same rate}. They \textit{cannot} be used in situations in which cash flows need to be discounted at different rates.

D. The Yield Curve and the Economy

1. Interest Rate Determination

Key issue: role of Fed and markets in determining interest rates

Note: With quantitative easing, the Fed attempted to influence long-term as well as short-term interest rates in 2008 through 2012

2. Interest Rate Expectations

Key issue: role of expectations in determining long-term interest rates

Note: The IRR is a complicated weighted average of the interest rates used to calculate the present values of the individual cash flows

Concept Check: all

5.4 Risk and Taxes

A. Risk and Interest Rates

Key issue: => relationship between interest rates and default risk
Notes:

1) Coca-cola recently issued bonds at lower rate than the U.S. government

=> suggests that Coca-cola has (or at least had) a lower chance of default than the U.S. government

2) long-term rates tend to exceed short-term rates (even for U.S. Treasuries)

=> suggests a risk-premium included in long-term rates

B. After-Tax Interest Rates

Key issues:

=> most interest income is taxable for investors and interest payments are tax deductible for corporations
=> taxes reduce the cash that investors get to keep from interest they earn
=> taxes reduce the cost of borrowing for firms
=> both investors and corporate borrowers care about the after-tax interest rate

After-tax interest rate: \( r_{AT} = r - (\tau \times r) = r(1 - \tau) \) \hspace{1cm} (5.8)

where:
\( r_{AT} = \) after-tax interest rate
\( r = \) before-tax interest rate
\( \tau = \) tax rate

Concept Check: all

5.5 The Opportunity Cost of Capital

Opportunity cost of capital (or simply cost of capital): best available expected return offered in the market on an investment of comparable risk and term to the cash flow being discounted.

Concept Check: all
5 Appendix

A. Discount Rates for a Continuously Compounded APR

Key issue: converting between APR and effective annual interest rate when interest is compounded continuously

Note: Can’t use equation 5.2 since $k = \infty$

$$r(1) = e^{APR} - 1$$  \hspace{1cm} (5A.1)

$$APR = \ln(1+r(1))$$  \hspace{1cm} (5A.2)

Khan Academy:

- [e and compound interest](https://www.khanacademy.org/math/precalculus/calculus/exponential-logarithmic-functions/exponential-growth-decay/a/comparing-e-and-re)
- [e as a limit](https://www.khanacademy.org/math/calculus-single-variable/exponential-logarithmic-functions/exponential-growth-decay/a/definition-of-e-as-a-limit)
- [Formula for continuously compounding interest](https://www.khanacademy.org/math/calculus-single-variable/exponential-logarithmic-functions/exponential-growth-decay/a/formula-for-continuously-compounding-interest)

Ex. Assume a bank pays an APR of 5% with continuous compounding. What is the effective annual interest rate?

$$r(1) = e^{.05} - 1 = .05127$$

Excel: = exp(.05) – 1

B. Continuously Arriving Cash Flows

Comment: You can skip this section