Chapter 4: The Time Value of Money

Fundamental question:

Problem: can't directly compare or combine cash flows at different points in time since they are not in the same units

Key => a dollar today does not have the same value as a dollar a year from today

Reason: you could invest the $1 today and have more than $1 in a year (as long as interest rates are positive)

Solution: convert all cash flows to equivalent values at a single point in time (so in the same units)

=> once have moved to a single point in time can compare or combine them

Ex. Would you trade $1 today for $2 in 20 years?

=> must first figure out:

a) What is the equivalent value in 20 years of $1 today?

or:

b) What is the equivalent value today of $2 received in 20 years?

Systematic steps to solving TVM problems:

1.

2.

- Perpetuity –

- Annuity –

- Single cash flow – one cash flow or a combination of single cash flows

3.

4. Repeat steps 2 and 3 as often as necessary to solve the problem

Note: This chapter includes a lot of equations. See Dr. Rich's TVM Appendix starting on p. 14 of these partial lecture notes for variable definitions and important notes on using the equations.

4.1 The Timeline

Notes:

1. Timelines are not required and not graded…but you will make many fewer mistakes if you use them
2. Today is year 0
3. A year from today is the end of the first year and the beginning of the second year; similarly, two years from today is the end of the second year and the beginning of the third year.
4. Inflows are positive while outflows are negative
5. I do all time lines in years
6. I vary size of tick marks to make timelines easier to read

Relative size of tick marks (from smallest to largest): monthly, semiannual, annual, 5-year, 10-year

Note: use as few "sizes" as possible

Timeline Examples:

Ex. Assume annual cash flows of $5 for years 3 through 9.

Ex. Assume monthly cash flows of $10 starting in year 1 and going through 1 year and 6 months.

Ex. Assume plan to deposit $175 into a bank account today and plan to take out $100 a year from today and to take out $110 two years from today.

4.2 The Three Rules of Time Travel

A. Rule 1: Comparing and Combining Values

Basic idea:

B. Rule 2: Moving Cash Flows Forward in Time

Basic idea:

Ex. Assume invest $1000 today at an interest rate of 4% per year. How much do you have at the end of a year?

=> Interest = balance in account × interest rate = $40 =

=> Balance in one year = beginning balance + interest = $1040 =

=> If the interest rate is 4% per year, the future value of $1000 in one year equals $1040.

Note: $1000 today and $1040 a year from today are equivalent if the interest rate is 4%

Ex. How much do you have at the end of two years?

=> Interest in second year = $41.60 =

Note: earn interest on not only original balance but on the balance at the end of the first year

=> in second year, earning interest on the interest earned in first year
=> earning interest on interest is called compounding

=> Balance in two years = balance at end of year 1 + interest earned during second year = $1081.60 =

Note: $1081.60 two years from today is equivalent to $1040 a year from today and $1000 today if the interest rate is 4%

1. Future value equations

Note: the book uses FV_n if we have calculated a future value of get the value n years from today and PV_n if we have calculated a present value to get the value n years from today. I prefer to just use V_n for the value n years from today…regardless of if calculated a present or future value to get the number.

Let:

C_n = cashflow that occurs n years from today
V_n = value n years from today
I_n = interest during year n
r_n = interest rate during year n

Notes:

1) C_0 = cashflow today
2) C can be positive (inflow) or negative (outflow)
3) V_0 = value today
4) $V_0 = C_0$: The value today of a cashflow today is the cashflow.
5) I_1 = interest earned during 1st year…between today (n = 0) to a year from today (n = 1).
6) For now we will assume r is the same for all years

$$V_1 = C_0 + I_1 = C_0 + r \times C_0 = C_0 \times (1+r)$$

Ex. Assume that today you deposit \$1000 at a 4% interest rate. How much is in your account a year from today?

V_1 = \$1040 =

Note: Can take the future value multiple times

$$V_2 = V_1 + I_2 = V_1 + r \times V_1 = V_1 \times (1+r) = C_0 \times (1+r) \times (1+r) = C_0 \times (1+r)^2$$

$$V_3 = V_2 + I_3 = V_2 + r \times V_2 = V_2 \times (1+r) = C_0 \times (1+r)^2 \times (1+r) = C_0 \times (1+r)^3$$

General equation: $V_n = C_0 \times (1+r)^n$ (4.1)

where: n = number of periods pushing cash flow into future

Ex. Assume you invest $1000 at 4% per year. How much do you have one year from today, two years from today, and 15 years from today?

V_1 = 1040 =

V_2 = 1081.60 =

V_{15} = 1800.94 =

C. Rule 3: Moving Cash Flows Back in Time

Basic idea (as mentioned in Chapter 3): to get a present value, back out interest could have earned if had the cash today

$V_0 = C_1 - I_1 = C_1 - r \times V_0$

=> $V_0 (1+r) = C_1$

=> $V_0 = \frac{C_1}{(1+r)} = C_1 \times \frac{1}{(1+r)} = C_1 \times (1+r)^{-1}$

Ex. You plan to deposit money in an account earning a 4% interest rate and would like to withdraw $1040 a year from today. How much do you need to deposit today?

$V_0 =$ 1000 =

Note: Obviously this makes sense given that previously we found that $1000 deposited today becomes $1040 a year from today if the interest rate is 4%.

Note: Can take present values multiple times

$$V_0 = V_1\left(\frac{1}{1+r}\right) = C_2\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right) = C_2\left(\frac{1}{1+r}\right)^2$$

$$V_0 = V_1\left(\frac{1}{1+r}\right) = V_2\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right) = C_3\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right) = C_3\left(\frac{1}{1+r}\right)^3$$

General Equation:

$$V_0 = C_n\left(\frac{1}{1+r}\right)^n = \frac{C}{(1+r)^n} = C(1+r)^{-n} \qquad (4.2)$$

Ex. How much need to deposit today into an account paying a 4% interest rate if want $1800.94 in account 15 years from today?

$V_0 = 1000 =$

Note: Again this makes sense given previously found that if deposit $1000 for 15 years at 4%, end up with $1800.94.

D. Applying the Rules of Time Travel

Key issues:

=> can combine or compare cash flows once convert to equivalent values at a single point in time
=> doesn't matter what point in time

4.3 Valuing a Stream of Cash Flows

Key: take present or future value to a single point in time then add them up

$$V_0 = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N} \quad (4.3)$$

$$V_0 = \sum_{n=0}^{N} \frac{C_n}{(1+r)^n} \quad (4.4)$$

Ex. Assume you plan to deposit funds into an account today that pays an interest rate of 6% per year. You plan to deposit an additional $100 into the account one year from today and withdraw $250 two years from today and $1000 five years from today. How much do you need to deposit today to make all the planned withdrawals?

$V_0 = -875.41766 =$

Or $V_0 = -875.41766 = 100(1.06)^{-1} - 250(1.06)^{-2} - 1000(1.06)^{-5}$ = 94.3396 – 222.4991 – 747.2582

=> planned future cash flows have value today of – $875.42
=> to fund withdrawals, need to deposit $875.42 today to offset negative value from withdrawals

Check:

$V_1 = 1027.94 =$

$V_2 = 839.62 =$

$V_3 = 890.00 =$

$V_4 = 943.40 =$

$V_5 = 0 =$

4.4 Calculating the Net Present Value

NPV = sum of the present value of all cash flows (positive and negative)

Notes:

1) you can use equation 4.4 above to calculate NPV
2) Excel's time value of money functions are quirky. So I prefer to build my own equations. If you use the functions correctly, you will get the same answer. At one point, my former students have told me they were not allowed to use the Excel functions when doing time value of money calculations on their jobs. I don't know if this is still the case.

4.5 Perpetuities and Annuities

A. Perpetuities

=> stream of equal cash flows that occur at regular intervals forever

Ex. Assume invest $100 at a 5% interest rate. How much can you withdraw each year forever?

$C_1 = 100 \times .05 = \$5$

Notes:

1) If withdraw the $5 of interest, $100 remains in the account. As a result, if interest rates do not change, you can withdraw $5 the following year, and the year after that, and every year thereafter forever.

2) Investing $100 at a 5% interest rate generates a perpetuity of $5 per year.

3) Can express result as equation:

$C_1 = V_0 \times r$

=> solving for V_0, get equation (4.7).

$$V_0 = \frac{C_1}{r} \qquad (4.7)$$

Ex. Assume want to invest enough that can withdraw $50 per year forever beginning a year from today. The account into which you are depositing money earns an interest rate of 4% per year. How much do you need to invest today to fund your withdrawals?

$V_0 = 1250 =$

Note: Interest earned per year = withdrawals = 50 =

B. Annuities

=> stream of equal cash flows that occur at regular intervals but which eventually stop

1. Present Value of an Annuity

=> textbook's derivation of the present value of an annuity equation is excellent.

=> however, another way to look at an annuity is as follows:

=> an annuity is the same as a perpetuity minus a perpetuity that begins after the annuity's final payment

Ex. 3 year annuity = (1) perpetuity minus (2) perpetuity with first payment 4 years from today

V_0 of (1) $= \frac{C}{r}$

V_0 of (2) $= \frac{C}{r}\left(\frac{1}{1+r}\right)^N$

Note: $V_N = \frac{C}{r}$ where N = number of cash flows in the annuity

$$=> V_0 = \frac{C}{r} - \frac{C}{r}\left(\frac{1}{1+r}\right)^N = \frac{C}{r}\left(1 - \left(\frac{1}{1+r}\right)^N\right) \qquad (4.9)$$

Ex. How much will you have to deposit today if want to withdraw \$100 per year for six years if earn an interest rate of 4% per year?

$V_0 = 524.214 =$

2. Future Value of an Annuity

=> books derivation of the equation is excellent

$$V_N = \frac{C}{r}\left((1+r)^N - 1\right) \qquad (4.10)$$

Ex. How much will you be able to withdraw six years from today if invest \$100 per year for six years (starting a year from today) and earn an interest rate of 4% per year?

$V_6 = 663.2975 =$

Note: Get same answer by taking future value of the \$524.214 got in previous example.

$V_6 = 663.2975 =$

C. Growing Cash Flows

1. Growing Perpetuities

Key => if want to make withdrawals forever where each payment grows by rate "g", account balance must grow at rate "g" as well.

=> this allows interest earned each period (and thus withdrawals) to grow at rate "g"
=> growth of account each year must be g × V_t
=> must redeposit part of the interest earn each period
=> amount must redeposit = g × V_t
=> only works if g < r
=> otherwise not enough interest to fund the growth

Ex. Assume deposit $1000 and want account balance to grow by 3% per year forever (so that interest earned and thus withdrawals can increase by 3% per year forever). What account balance will you have at the end of the next 3 years?

=> V_1 = 1030 =

=> V_2 = 1060.9 =

=> V_3 = 1092.927 =

Q: What can you withdraw from account each year?

=> whatever you earn in excess of 3%.

=> assume earning 7% on account, can withdraw (.07 - .03) × beginning account balance => can withdraw 4% of beginning account balance.

=> C_1 = 40 =

=> C_2 = 41.20 =

Note: 41.20 is 3% larger than $40

=> C_3 = 42.436 =

Note: 42.436 is 3% larger than $41.20

Check:

V_1 = 1030 =

V_2 = 1060.9 =

V_3 = 1027.727 =

General equation for withdrawals: $C_1 = V_0(r - g)$

$$=> V_0 = \frac{C_1}{r-g} \tag{4.11}$$

Ex. How much do you need to deposit today in an account earning a 7% interest rate if you want to withdraw $40 a year from today and would like for subsequent withdrawals by increase by 3% each? (Note: Know must be 1000 from example on p. 10).

$V_0 = 1000 =$

2. Growing Annuities

a. Present values

Can look at a growing annuity the same way we looked an annuity a few pages back=> a growing annuity is the same as a growing perpetuity minus a growing perpetuity after the growing annuity stops.

Ex. 3 year growing annuity = (1) a growing perpetuity – (2) a growing perpetuity with first payment 4 years from today

V_0 of (1) $= \frac{C}{r-g}$

V_0 of (2) $= \frac{C(1+g)^N}{r-g}\left(\frac{1}{1+r}\right)^N$

Note: $V_N = \frac{C}{r}$ where N = number of years in annuity

$$=> V_0 = \frac{C}{r-g} - \frac{C(1+g)^N}{r-g}\left(\frac{1}{1+r}\right)^N = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^N\right) \tag{4.12}$$

Ex. How much do you need to deposit today if want to make three annual withdrawals where the first withdrawal equals \$100 one year from today and subsequent withdrawals grow by 1% each? Assume you earn an interest rate of 4%?

$V_0 = 280.22 =$

Ex. How much will you have to deposit today if want to withdraw \$100 a year from today and want subsequent withdrawals to grow by 1% each through six years from today if you earn an interest rate of 4% per year?

$V_0 = 536.884 =$

Note: Larger 280.22 in previous example where only 3 withdrawals and larger than \$524.21 in the p. 9 example where all withdrawals were \$100.

=> deposit needs to be larger since withdrawals (after first one) are larger.

b. Future Values

$$V_t = \frac{C}{r-g}\left((1+r)^N - (1+g)^N\right) \qquad (4.C)$$

Note: Can get equation treating equation (4.12) as C_0 (the amount of cash today that is equivalent to the growing annuity) in (4.1) and simplifying.

=> taking future value of present value of growing annuity equation

$$V_N = \left(\frac{C}{r-g}\left(1-\left(\frac{1+g}{1+r}\right)^N\right)\right)(1+r)^N = \frac{C}{r-g}\left((1+r)^N - (1+g)^N\right)$$

Note: Essentially calculating the value at some future date of a perpetuity that starts a year from today. But subtracting away the value of all payments that occur after the annuity payments stop.

Ex. Assume you deposit \$100 a year from today and make subsequent deposits that grow by 1% each through six years from today. How much will you have in your account six years from today? Assume also that earn 4% on the account.

$V_6 = 679.33 =$

Note: Same result if take future value of previous example.

$V_6 = 679.33 =$

4.6 Using an Annuity Spreadsheet or Calculator

Notes:

1) I am more concerned that you know how to build a spreadsheet than that you know how to use one created the authors of the text
2) The authors' spreadsheet is well-done

4.7 Non-Annual Cash Flows

Key ideas:

=> can use the equations in this chapter for cash flows that occur at any interval
=> must have interest rate that matches frequency of cash flows
  Ex. if quarterly cash flows, must use quarterly interest rate
  Ex. if semiannual cash flows, must use semiannual interest rate

4.8 Solving for the Cash Payments

Key idea (for section 4.8, 4.9, and the appendix): can solve for any one unknown variable in the time-value-of-money equations

4.9 The Internal Rate of Return

Internal Rate of Return: interest rate that sets net present value of cash flows equal to zero.

=> essentially solving for "r" in any of the equations in this chapter.

4.Appendix

Key idea: can solve for number of payments (in annuity) or number of periods (with single cash flow) if it is the one unknown variable.

4.A Dr. Rich's TVM Appendix

A. You really only need four equations to do everything in Chapter 4

Future (or present) value of single cash flow: $V_t = C \times (1+r)^n$ (4.1)

Present value of a (growing) perpetuity: $V_0 = \frac{C_1}{r-g}$ (4.11)

Present value of a (growing) annuity: $V_t = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^N\right)$ (4.12)

Future value of a (growing) annuity: $V_t = \frac{C}{r-g}\left((1+r)^N - (1+g)^N\right)$ (4.C)

B. Notes on time-value-of-money equations (these are really important)

1) "n" equals number of periods moving a single cash flow to right or left on timeline

=> "- n" moves us n periods to the left on the timeline

Ex. Calculate the value today of $100 received two years from today if the interest rate equals 4%.

$V_0 = 92.456 =$

2) C = 1st cash flow in series (may be only cash flow)
3) r = interest rate
4) g = growth rate of cash flows (after 1st one) in a series of cash flows
Note: g = 0 for constant cash flows and g < 0 for shrinking cash flows

5) N = number of cash flows in a series of cash flows
6) Can solve for any one unknown variable if plug in numbers for all the others
=> "V_t" may be one of the things we know.

7) Where end up on time-line:

a) PV of single cash flows: n periods before the cash flow
b) FV of single cash flows: n periods after the cash flow
c) PV of a series (growing annuity or growing perpetuity): one period before first cash flow
Note: period defined by time between cash flows
=> if monthly cash flows find value one month before first cash flow
=> if annual cash flows find value one year before first cash flow
=> if quarterly cash flows (every 3 months) find value one quarter (3 months) before first cash flow
d) FV of series (growing annuity): date of last cash flow

8) I add a "t" subscript to V keep track of where I end up on the timeline

9) All of the equations can be used for any frequency of cash flows (annual, monthly, daily, etc.)

a) When calculating the PV or FV of a single cash flow, can use an interest rate for any period of time

=> be careful to move correct number of periods

Ex. If want future value in two years, n = 2 if using an annual rate but n = 24 if using a monthly rate (since 2 years is 24 months)

b) When calculating the PV or FV of a series of cash flows, must use the interest rate that matches the time between the cash flows.

Ex. monthly cash flows => must use monthly interest rate
Ex. semiannual cash flows => must use semiannual interest rate
Ex. annual cash flows => must use an annual interest rate

C. Harder examples

1. Upon retirement 35 years from today, you would like to make your first of 15 annual withdrawals from your savings account. You would like for the first withdrawal to be $180,000 and would like to be able to increase your withdrawals by 5% per year in order to allow for inflation. How much would you have to deposit today to achieve your goal if your account pays 8.5% per year? What is the size of your second withdrawal and your final withdrawal?

   Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

   Deposit today:

   $V_t = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^N\right)$

   $V_{34} = 1{,}998{,}022.07 =$

   Steps: 2) pattern, 3) equation, 4) fill in

   $V_t = \frac{C_n}{(1+r)^n} = C_n(1+r)^{-n}$

   $V_0 = 124{,}735.24 =$

   Second withdrawal:

   Steps: 1) timeline, 2) pattern, 3) equation, 4) fill in

   $V_t = C \times (1+r)^n$

   $C_{36} = 189{,}000 =$

   Final withdrawal:

   Steps: 1) timeline, 2) pattern, 3) equation Q: PV or FV?

   $V_t = C \times (1+r)^n$

   $C_{49} = 356{,}387.69 =$

2. Assume you plan to deposit $100 into a savings account 5 years from today. Thereafter you plan to make annual deposits that grow by 3% per year. How much is in your account after you have made 7 total deposits if the account pays 9% per year? What is size of 7th deposit?

   Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

   $V_t = \frac{C}{r-g}((1+r)^N - (1+g)^N)$

   $V_{11} = 996.94 =$

   Steps: 2) pattern of cash flow, 3) write down equation; Q: PV or FV?

   $V_t = C \times (1+r)^n$

   $C_{11} =$

3. Assume you deposit $1000 in an account that earns an interest rate of 6% per year. You plan to make annual withdrawals from your account beginning four years from today. Your plan for your first withdrawal to equal $55 and for your subsequent withdrawals to continue forever. At what rate can your withdrawals grow?

   Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

   $V_t = C \times (1+r)^n$

   $V_3 = 1191.02 =$

   $V_t = \frac{C}{r-g}$

   => g =

4. Assume that you have just deposited \$500 into an account that pays an interest rate of 8% per year. You plan to make equal annual withdrawals from this account with the first withdrawal coming 5 months from today and the final withdrawal coming 5 years and 5 months from today. How large can you make each of your withdrawals?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

$$V_t = \frac{C}{r-g}\left(1-\left(\frac{1+g}{1+r}\right)^N\right)$$

$$V_t = \frac{C}{.08}\left(1-\left(\frac{1}{1.08}\right)^6\right)$$

Q: Where end up on the timeline when take present value of the withdrawals?
=> 7 months ago
=> need to figure out equivalent 7 months ago of \$500 today (so can solve for C)

Steps: 2) pattern, equation; Q: PV or FV?

$$V_t = \frac{C_n}{(1+r)^n}$$

$V_{-7mo} = 478.05 =$

$$V_t = \frac{C}{r-g}\left(1-\left(\frac{1+g}{1+r}\right)^N\right)$$

$V_{-7mo} = 478.05 =$

$=> C =$