Chapter 5: Interest Rates

Big Picture: Cash flows usually occur more than once per year and interest usually compounds more than once per year. We have to adjust for these.

I. Interest Rate Quotes and the Time Value of Money

A. Key ideas

1. Compounding: earn interest on interest because interest is added to the balance.

2. Interest rates typically quoted in one of two basic ways:

   a. Annual Percentage Rates [APR] – annual interest rate that ignores the impact of compounding
   b. Effective interest rate \([r(t)]\) – interest rate that includes the impact of compounding

   \(t = \text{time frame of the interest rate in years}\)
   \(\Rightarrow \text{actual interest rate per period} \ t\)

   Ex.
   \[r\left(\frac{1}{12}\right) = \text{effective monthly rate}\]
   \[r(1) = \text{effective annual interest rate}\]

   Note: \(r(1)\) is also called 1) the APY (Annual Percentage Yield) because of the Truth in Savings Act and 2) the EAR (effective annual rate).

   Ex. Assume given two interest rates for an account. The APR is 6\% and the APY is 6.17\%.
   \(\Rightarrow\) if deposit $100 for a year, end up with $106.17 not $106.

3. With the exception of continuous compounding (not covered in this class), use only effective interest rates in time value of money calculations

4. When calculating the PV or FV of a single cash flow, can use any effective rate

5. When calculating the PV or FV of a series of cash flows, must use the effective rate that matches the time between the cash flows.

   Ex. monthly cash flows \(\Rightarrow\) must use effective monthly rate
   Ex. semiannual cash flows \(\Rightarrow\) must use effective semiannual rate

Note: You can click on most timelines to view a video clip explaining the timeline
B. Converting interest rates

1. Converting APRs to effective rates

\[
r(t) = \frac{APR}{k}
\]  
\[(5.2)\]

where:

k = number of compounding periods per year

\[t = \text{time frame of the interest rate in years} = \frac{1}{k}\]

Note: any time given an APR, must start with this equation

2. Converting between effective interest rates for different time periods

\[
r(t) = (1 + r)^n - 1
\]  
\[(5.1)\]

Usefulness: convert to an effective rate that matches the time between cash flows

Notes:

1) n = conversion ratio
2) to convert to a longer period, n > 1
3) to convert to a shorter period, n < 1

Ex. If want an interest rate for a period that is twice as long as the one you start with, n = 2

Ex. If want an interest rate for a period that is twelve times as long as the one you start with, n = 12

Ex. If want an interest rate for a period that is one-fourth as long as the one you start with, n = 1/4
C. Examples

1. Assume an APR of 6% per year with semiannual compounding. What is the effective annual interest rate and the effective monthly interest rate on this account?

\[
r(t) = \frac{APR}{k}
\]

\[
r\left(\frac{1}{2}\right) = \frac{0.06}{2} = 0.03
\]

=> effective semiannual rate (half a year) is 3%

\[
r(t) = (1 + r)^n - 1
\]

\[
r(1) = (1.03)^2 - 1 = 0.0609
\]

\[
r\left(\frac{1}{12}\right) = (1.03)^{\frac{1}{6}} - 1 = 0.004939
\]

Note: \( r\left(\frac{1}{2}\right) = 0.03 \), \( r(1) = 0.0609 \), and \( r\left(\frac{1}{12}\right) = 0.004939 \) are equivalent

=> end up with same amount of money at the end

Ex. If invest $100 for a year, then your account balance at the end of the year equals:

\[
FV_1 = 100(1.03)^2 = 100(1.0609) = 100(1.004939)^{\frac{1}{2}} = 106.09
\]
2. Eight months from today you want to make the first of 14 quarterly withdrawals from a bank account. Your first withdrawal will equal $10,000 and each subsequent withdrawal will grow by 1% each. How much do you need to deposit today if the account pays an APR of 9% with monthly compounding?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ r(t) = \frac{APR}{k} \]

\[ r(\frac{1}{12}) = \frac{.09}{12} = .0075 \]

\[ r(t) = (1 + r)^n - 1 \]

\[ r(\frac{1}{4}) = (1.0075)^3 - 1 = .022669 \]

\[ PV_t = \frac{C}{r-g} \left[ 1 - \left( \frac{1+g}{1+r} \right)^N \right] \]

\[ PV_{5mo} = \frac{10,000}{.022669-.01} \left[ 1 - \left( \frac{1.01}{1.022669} \right)^{14} \right] = 126,401.27 \]

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ PV_t = \frac{C_n}{(1+r)^n} \]

\[ PV_0 = \frac{126,401.27}{(1.0075)^{14}} = 121,766.03 \]
3. What if you want to make the first withdrawal one month from today (and nothing else changes)?

Q: Will the amount you deposit be larger or smaller if 1\textsuperscript{st} withdrawal is in one month instead of eight months? Larger

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ r\left(\frac{1}{\sqrt{12}}\right) = .075 \quad ; \quad r\left(\frac{1}{4}\right) = .02269 \quad ; \quad PV_{-2mo} = 126,401.27 \]

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ FV_t = C \times (1 + r)^n \]

\[ FV_0 = 126,401.27 \times (1.0075)^2 = 128,304.39 \]
4. A bond matures for $1000 three years and ten months from today. The annual coupon on the bond equals $60 but coupons are paid semiannually. What is the value of the bond if it earns a return of 8% per year?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) fill in known variables and solve for unknown variable, 6) make sure know where end up on timeline.

\[ r(t) = (1 + r)^n - 1 \]
\[ r(\frac{1}{2}) = (1.08)^{1/2} - 1 = 0.03923 \]

Coupons:

\[ PV_t = \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r}\right)^N\right) \]
\[ PV_{2\text{mo}} = \frac{30}{0.03923} \left(1 - \left(\frac{1}{1.03923}\right)^8\right) = 202.6257 \]

\[ FV_t = C \times (1+r)^n \]
\[ FV_0 = 202.6257 \times (1.08)^{2/12} = 205.2415 \]

Par

\[ PV_t = \frac{C_n}{(1+r)^n} \]
\[ PV_0 = \frac{1000}{(1.08)^{12}} = 744.5187 \]

Price = 205.25 + 744.52 = 949.76

Calculator:

PV_{2\text{mo}}: 30 = PMT, 1000 = FV, 8 = N, 3.923 = I% \Rightarrow PV = 937.6555

FV_0: 937.6555 = PV, 8 = I%, 2/12 = N \Rightarrow FV = 949.76
D. Practicing N and n

1. Assume you want to take out a loan and can afford to make monthly payments of $100 each starting five months from today that will continue through three years and three months from today. Assume you want to determine the amount you can borrow. Which equation(s) would you use? What would you use for “N” or “n” in each equation?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.
A: 1) present value of annuity, N = 35; 2) present value of single cash flow, n = 4

2. Assume your lender will allow you to pay off your loan (in number 1 above) by making payments of only $10 per month (still starting five months from today and continuing through three years and three months from today). You would then make a balloon payment three and a half years from today to pay off the balance of the loan. Assume you want to determine the size of the balloon payment. Which equation(s) would you use? What would you use for “N” or “n” in each equation?

Steps: 1) timeline, 2) pattern of cash flows, 3) present or future value, 4) equation or equations, 5) figure out “N” or “n”.
A: 1) future value of annuity ($90 payments), N = 35; 2) future value of a single cash flow, n = 3

II. Determinants of interest rates

A. Inflation

Nominal interest rate: growth rate of money
Real interest rate: growth rate of purchasing power

Ex. Assume the nominal interest rate is 6% per year and that the real interest rate is 4% per year

=> after one year you will:

1) have 6% more dollars
2) be able to buy 4% more stuff

1. Basic idea: investors care about real rather than nominal interest rates

2. if investors expect inflation to increase, nominal rates will increase

=> compensates investors for their loss of purchasing power
3. Converting between nominal and real interest rates

\[ r_r = \frac{r - i}{1 + i} \quad (5.5) \]

where:

- \( r \) = nominal interest rate
- \( i \) = inflation rate
- \( r_r \) = real interest rate

Note: can use expected or realized rates

Ex. Assume that the nominal interest rate is 6% per year and that inflation is 5% per year. What is the real interest rate?

\[ r_r = 0.00952 = \frac{0.06 - 0.05}{1.05} \]

Note: the difference between the nominal rate and the inflation rate is a pretty good approximation of the real rate if inflation is low.

B. The Fed

Basic idea: The Federal Reserve lowers or raises interest rates to stimulate or cool off the economy.

Key: if lower interest rates, more investments worthwhile since NPVs rise
C. Maturity

Basic ideas:

1) interest rates vary by maturity
   Ex. can see how interest rates on U.S. Treasuries vary by maturity at Yahoo Finance.
   http://finance.yahoo.com
   Links to follow: Investing; Bonds

   Note: credit default swap prices now indicate a nonzero chance of default by the U.S. Treasury

2) long-term rates reflect what investors expect will happen to short-term rates in the future

3) long-term rates usually exceed short-term rates

D. Taxes

After-tax interest rate: \( r_{AT} = r - (\tau \times r) = r(1 - \tau) \) \hspace{1cm} (5.8)

Where:
\( r_{AT} \) = after-tax interest rate
\( r \) = before-tax interest rate
\( \tau \) = tax rate

Basic idea: investors care about after-tax returns

=> the higher the tax rates, the higher the return investors will demand

E. Risk

Basic idea: the greater the risk, the higher the interest rate