Instructions: Answer each question thoroughly. All answers must be legible. Please indicate your solution by circling the answer.

1. (20 points) The consumer seeks to maximize the following utility function

\[ U(x, y) = -y^{-1} - x^{-1} \]  

subject to the budget constraint \( I = P_x X + P_y Y \).

\( \checkmark \) (a) Calculate Marshallian demand for \( x(P_x, P_y, I) \). Rigorously show the effect of an increase in the price of \( x \) on demand for \( x \).

\( \checkmark \) (b) It is common to hear someone say that money cannot buy love or happiness,\(^1\) but what about utility? Rigorously show the effect of an increase in exogenous income on indirect utility using the utility function provided above.

\( \checkmark \) 2. (20 points) Suppose that a consumer has the following expenditure function:

\[ E(P_x, P_y, V) = P_y V + 2P_x^{\frac{1}{2}} P_y^c \]  

\( \checkmark \) (a) What is the value of the parameter, \( c \)? Explain your answer.

\( \checkmark \) (b) Rewrite the expenditure function with the value of \( c \) found in part a. Use Shepherd's Lemma to derive the compensated demand for \( y^c(P_x, P_y, V) \).\(^2\) Rigorously show the effect of a change in the price of \( x \) on demand. Are these goods net complements, net substitutes, gross complements or gross substitutes?

3. (15 points) A consumer has Cobb-Douglas utility, \( U = f(x, y) = (xy)^{\frac{1}{2}} \). Each good can only be purchased in markets at prices, \( P_x \) and \( P_y \), using exogenous (non-labor) income, \( I \). Assume that (i) \( x, y > 0 \); (ii) \( P_x = 1, P_y = 4, I = 8 \).\(^3\) The demand functions are provided for you: \( x(P_x, P_y, I) = \frac{I}{2P_x} \) and \( y(P_x, P_y, I) = \frac{I}{2P_y} \). The government wishes to collect tax revenue and is choosing between a "per-unit tax" (excise tax) or a lump-sum income tax.

\( \checkmark \) (a) Calculate the demand for \( x \) and \( y \) if the government taxes each unit of \( x \) at 1.

\( \checkmark \) (b) Assume government taxes non-labor income, \( I \), by an amount equalling the tax revenue collected under the excise tax from the previous question. Calculate demand for \( x \) and \( y \) under this lump-sum income tax.

\( \checkmark \) (c) Which tax regime does the consumer prefer? Why?

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\( ^1 \)The Beatles wrote, "I don't care too much for money / For money can't buy me love".

\( ^2 \)You are not required to prove Shepherd's Lemma.

\( ^3 \)All prices and income are in US dollars for simplicity. There are no externalities or leisure in this problem, either.
Let \( u(x, y) = -\frac{1}{y} - \frac{1}{x} \)

\[(a) \quad \frac{d^2 x}{dx^2} = -\frac{1}{y} - \frac{1}{x} + \lambda \left( I - P_x x - P_y y \right) \]

\[\text{FOC1:} \quad \frac{dx}{dx} = \frac{1}{x^2}, \quad \lambda P_x = 0 \]

\[\lambda = \frac{1}{P_x x^2} = \frac{1}{P_y y^2} \]

\[\text{FOC2:} \quad \frac{dy}{dx} = -\frac{1}{y^2}, \quad P_x X^2 = P_y Y^2 \]

\[\frac{X^2}{P_x} = \frac{Y^2}{P_y} \]

\[X = \left( \frac{P_y}{P_x} \right)^{1/2} Y \]

Substitute \( X \) into FOC3.

\[I - P_y Y - P_x X = 0 \]

\[I = P_y Y + P_x \left[ \left( \frac{P_y}{P_x} \right)^{1/2} Y \right] \]

\[X^*(P_x, P_y, I) = \left( \frac{P_y}{P_x} \right)^{1/2} \left( \frac{I}{P_y + (P_x P_y)^{1/2}} \right) \]

\[y^*(P_x, P_y, I) = \frac{I}{P_y + (P_x P_y)^{1/2}} \]

\[\left( \frac{X^*(P_x, P_y, I)}{P_x + (P_x P_y)^{1/2}} \right)^2 \left[ 1 + \left( \frac{P_x}{P_y} \right)^{1/2} \right] < 0 \]

\[\frac{dX}{dP_x} \left. \frac{d}{dP_x} \right] + \frac{I}{\left( P_x + (P_x P_y)^{1/2} \right)^2}, \quad \left[ 1 + \left( \frac{P_x}{P_y} \right)^{1/2} \right] < 0 \]

\[\frac{dX}{dP_x} < 0\]
1. (b) Find indirect utility.

\[ u(x^*, y^*, I) = \]

\[ \begin{align*}
V &= \frac{1}{x^*} - \frac{1}{y^*} \\
&= \frac{1}{I} - \frac{1}{P_x + (p_x p_y)^{1/2}}
\end{align*} \]

\[ \begin{align*}
V &= - \frac{P_x - (p_x p_y)^{1/2} - P_y - (p_x p_y)^{1/2}}{I} \\
&= \frac{1}{I} [P_x + 2(p_x p_y)^{1/2} + P_y]
\end{align*} \]

\[ \frac{\partial V}{\partial I} = \left[ \frac{P_x + 2(p_x p_y)^{1/2} + P_y}{I^2} \right] > 0 \]

An increase in non-labor income causes an increase in indirect utility.

\[ \frac{\partial V}{\partial I} > 0 \]
2. \[ E(P_x, P_y, \bar{V}) = P_y \bar{u} + 2 P_x^{1/2} P_y^{1/2} \]

9. Expenditure functions are homogenous of degree one \((H^1)\) in all prices. Therefore \(c = 1/2\).

\[ E(tP_x, tP_y, \bar{u}) = tP_y \bar{u} + 2(tP_x)^{1/2} (tP_y)^{1/2} \]

\[ = tP_y \bar{u} + t \frac{1}{2} P_x^{1/2} P_y^{1/2} \]

\[ = t \left( P_y \bar{u} + (P_x P_y)^{1/2} \right) \]

\[ E(tP_x, tP_y, \bar{u}) = t E(P_x, P_y, \bar{u}) \]

b. \( E(P_x, P_y, \bar{u}) = P_y \bar{u} + 2(P_x P_y)^{1/2} \)

Shepherd's lemma: \[ \frac{dE(P_x, P_y, \bar{u})}{dP_x} = \chi_c(P_x, P_y, \bar{u}) \]

\[ \frac{dE(P_x, P_y, \bar{u})}{dP_x} = \frac{1}{2} P_x^{1/2} P_y^{1/2} \quad \frac{dE(P_x, P_y, \bar{u})}{dP_y} = \bar{u} + \frac{1}{2} P_x^{1/2} P_y^{1/2} \]

\[ \chi^c(P_x, P_y, \bar{u}) = \left( \frac{P_y \bar{u}}{P_x} \right)^{1/2} \]

\[ \frac{d\chi^c(P_x, P_y, \bar{u})}{dP_x} = \frac{1}{2} \frac{P_y^{1/2}}{P_x^{3/2}} < 0 \]

\[ \frac{d\chi^c(P_x, P_y, \bar{u})}{dP_y} = \frac{1}{2} \frac{P_x^{1/2}}{(P_x P_y)^{3/2}} > 0 \]

Next substituted
$U = f(x, y) = (xy)^{\frac{1}{2}}$

$X(p_x, p_y, I) = \frac{I}{2p_x}$

$y(p_x, p_y, I) = \frac{I}{2p_y}$

Baseline (no tax),

$$\max u = V(p_x, p_y, I) = \left[ \frac{I}{2p_x} \right]^{\frac{1}{2}} \left[ \frac{I}{2p_y} \right]^{\frac{1}{2}}$$

$$V(1, 4, 8) = \frac{8}{2(1.4)^{1/2}} = \frac{4}{2} = 2\text{ u.t.}'s$$

$$V(1, 4, 8) = 2\text{ u.t.}'s \quad \text{(no tax)}$$

(a) Exercise for $x$ ($t = 1$ per unit of $x$ consumed)

$p_x + t = \tilde{p}_x$

$\tilde{p}_x = 2$

$$X(\tilde{p}_x, p_y, I) = \frac{I}{2\tilde{p}_x} = \frac{8}{2.2} = 3.636\text{ u.t.}'s \text{ of } x$$

$p_y = 4$

$$y(\tilde{p}_x, p_y, I) = \frac{I}{2p_y} = \frac{8}{2.4} = 1\text{ u.t.}'s \text{ of } y$$

$$V(\tilde{p}_x, p_y, I) = \frac{I}{2(\tilde{p}_x p_y)^{1/2}} = \frac{8}{2(2.4)^{1/2}} = \frac{4}{2.2^{1/2}} = \frac{2}{2^{1/2}}$$

$2^{1/2} = 1.4$

$$V = 2 \div 1.4 = 1.43\text{ u.t.}'s$$

$$V(\tilde{p}_x, p_y, I) = 1.43\text{ u.t.}'s$$
3. (b) Lump-Sum Tax

First find tax revenue collected in part 3a.

\[\text{tax revenue (TR)} = \sum X (P_x, P_y, I)\]

\[= 81 \cdot 2 \text{ units demanded}\]

\[TR = 82 \text{ in total tax revenue}\]

Second, set tax on income at TR = 82. Therefore,

\[\frac{I}{2} = 82 - 2 = 80. \text{ } P_x \text{ and } P_y \text{ are 1 and 2.}\]

\[\frac{\mathcal{I}}{2} = 6\]

\[\left[\begin{array}{c}
\chi (P_x, P_y, \mathcal{I}) = \frac{\overline{I}}{2P_x} = \frac{6}{2 \cdot 1} = 3 \text{ units of GX}
\end{array}\right]\]

\[\left[\begin{array}{c}
y (P_x, P_y, \mathcal{I}) = \frac{\mathcal{I}}{2P_y} = \frac{6}{2 \cdot 4} = 3 \text{ units of GY}
\end{array}\right]\]

\[V (P_x, P_y, \mathcal{I}) = \frac{\overline{I}}{2(P_x P_y)^{1/2}}\]

\[= \frac{6}{2(1.4)^{1/2}}\]

\[= \frac{3}{2}\]

\[V(P_x, P_y, I) = 1.5 \text{ utils}\]
3. (c),

<table>
<thead>
<tr>
<th>tax regime</th>
<th>$V(p_x, p_y, I)$</th>
<th>Demand X</th>
<th>Demand Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>baseline</td>
<td>2 u.t.5</td>
<td>4 units</td>
<td>1 unit</td>
</tr>
<tr>
<td>excise tax</td>
<td>1.43 u.t.5</td>
<td>2 units</td>
<td>1 unit</td>
</tr>
<tr>
<td>lump sum</td>
<td>1.5 u.t.5</td>
<td>3 units</td>
<td>3/4 units</td>
</tr>
</tbody>
</table>

Consumer prefers lump sum tax regime to excise tax regime bloc lump sum indirect utility is larger than excise tax (1.5 > 1.43).
4. Only equations 4 and 5 are demand equations. Marshellian demand equations are homogeneous of degree zero ($H^0$) in all prices and income. Only 4 and 5 are $H^0$ in all prices and income.

\[
3. y(t_P, t_P, t_I) = \frac{t_I}{2t_P + t_P} = \frac{t_I}{2t_P} = \frac{t_I}{2t_P (p_p + p_p)} = \frac{t_I}{(p_p + p_p)(p_p + p_p)} = \frac{t_I}{p_p(p_p + p_p)}
\]

\[
= \frac{1}{2} \cdot y(p_p, p_p)
\]

\[
4. y(t_P, t_P, t_I) = \frac{t_I}{(t_P + t_P)(t_P + t_P) + t_I} = \frac{t_I}{(t_P + t_P)^2 + t_I} = \frac{t_I}{(t_P)^2 + t_I}
\]

\[
= \frac{t_I}{(t_P)^2 + t_I} = \frac{t_I}{(p_p)^2 + t_I} = \frac{t_I}{p_p(p_p + t_I)} = y(p_p, p_p, t_I)
\]

\[
5. y(t_P, t_p, t_I) = \frac{t_I + t_P - t_P}{2t_P}
\]

\[
= \frac{t_I}{2t_P} = \frac{t_I}{2p_p} = y(p_p, p_p, t_I)
\]
5. If, in a model with two goods, \( x \) and \( y \), if \( x \) is inferior, then \( x \) and \( y \) are gross complements in response to an increase in the price of \( y \). \[ \text{FALSE} \]

\[
X(P_x, P_y, E(P_x, P_y, u)) = X^c(P_x, P_y, u) \]

\[
\frac{dx}{dP_y} + \frac{dx}{dE} \frac{dE}{dP_y} = \frac{dx^c}{dP_y}
\]

\[
\frac{dx}{dP_y} = \frac{dx^c}{dP_y} - \frac{dx}{dE} \frac{dE}{dP_y}
\]

\[
\frac{dx}{dP_y} = \frac{dx^c}{dP_y} - y^c(P_x, P_y, u) \frac{dx}{dE}
\]

If we assume continuous, transitive and complete preferences, then \( x \) will have diminishing MRS under quasi-concave utility functions. As a result, indifference curves in 2-dimensional space are convex, which means that \( \frac{dx^c}{dP_y} > 0 \).

This is simply a corollary of Hicks' 1st Law. The demand function, \( y^c > 0 \), and we are given \( \frac{dx}{dP_y} < 0 \). Therefore, we can sign the Slutsky partial derivative, \( \frac{dx}{dP_y} \), using the Slutsky equation:

\[
\frac{dx}{dP_y} = (+) + (-)(+)(-)
\]

\[
\frac{dx}{dP_y} = (+) + (+) > 0
\]

Or, \( \frac{dx}{dP_y} > 0 \). If \( \frac{dx}{dP_y} > 0 \), \( x \) and \( y \) are gross substitutes.
According to Hicks, most goods in the economy are substitutes. [TRUE]

Use the compensated demand function with n-goods and apply Euler's theorem. As compensated demand is homogeneous of degree zero in all prices, this is:

\[ \chi_c(P_1, P_2, \ldots, P_n, \bar{u}) \]

\[ \frac{\partial \chi_c}{\partial P_1} P_1 + \frac{\partial \chi_c}{\partial P_2} P_2 + \cdots + \frac{\partial \chi_c}{\partial P_n} P_n = 0 \quad (Euler) \]

\[ \frac{\partial \chi_c}{\partial P_1} \chi_c + \frac{\partial \chi_c}{\partial P_2} \chi_c + \cdots + \frac{\partial \chi_c}{\partial P_n} \chi_c = 0 \quad (\text{Div. by } \chi_c) \]

\[ e^{c_{1, P1}} + e^{c_{1, P2}} + \cdots + e^{c_{1, Pn}} = 0 \quad (elasticity definition) \]

Rewrite: \[ e^{c_{1, P2}} + \cdots + e^{c_{1, Pn}} \approx \sum_{i=2}^{n} e^{c_{i, P1}} \]

\[ e^{c_{1, P1}} + \sum_{i=2}^{n} e^{c_{i, P1}} = 0 \]

If \( e^{c_{i, P1}} \leq 0 \), then homogeneity of compensated demand implies \( \sum_{i=2}^{n} e^{c_{i, P1}} = 0 \), which means "most goods are substitutes."
6. If \( \frac{dx}{dt} > 0 \) (given), then according to Sultky:

\[
\frac{dx}{dt} = \frac{dx^c}{dp_1} - x^c(p_1, p_2, \bar{u}) \frac{dx_1}{dt} < 0
\]

\( \frac{dx^c}{dp_1} \geq \frac{dx_1}{dt} \)

\( \frac{dx_1}{dt} < 0 \)

Show that \( \frac{dx_1}{dp_1} \) and \( \frac{dx^c}{dp_1} \) have opposite signs.

(i) Composited demand is \( H_0 \) in all prices. Gours's theorem:

\[
\frac{dx^c}{dp_1} + \frac{dx_1}{dp_1} \frac{2x^c}{dx_1} p_2 = 0
\]

\[
\frac{dx^c}{dp_1} + \frac{dx_1}{dp_1} \frac{2x^c}{dx_1} p_2 = 0
\]

\[
\frac{dx^c}{dp_1} + \frac{dx_1}{dp_1} \frac{2x^c}{dx_1} p_2 = 0
\]

\[
e_c p_1 + e_{x_1} p_2 = 0
\]

\[
e_{x_1} p_1 = -e_{x_2} p_2
\]

By assumption, \( e_{x_2} p_2 < 0 \) due to curvature of indifference curve. Therefore \(-e_{x_2} p_2 > 0\) which requires \( e_{x_1} p_1 < 0 \). This therefore means that \( \frac{dx_2}{dp_1} > 0 \).

(ii) If \( \frac{dx_1}{dt} > 0 \), then \( \frac{dx_1}{dt} < 0 \) ("inferior"). Use Cost Not Aggregation

\[
\text{to examine the sign of} \ \frac{dx_2}{dp_1} \text{ Diff. budget constant}
\]

\( \text{using demand function} \ x_1(p_1, p_2, I) \text{ and } x_2(p_1, p_2, I) \text{ with respect to } p_1. \)
6. (cont.)

\[ P_1 x_1 (P_1, P_2, I) + P_2 x_2 (P_1, P_2, I) = 0 \]

differentiate \( P_2 \),

\[ \frac{P_2 x_1}{dP_1} + \frac{x_1}{dP_1} + \frac{P_2 x_2}{dP_1} = 0 \]

Convert to Constant Elasticity expression:

\[ D_1 e_{1, P_1} + D_1 + D_2 e_{2, P_1} = 0 \]

Rewrite as \( e_{2, P_1} \):

\[ D_2 e_{2, P_1} = - (D_1 + D_1 e_{1, P_1}) \]

\[ D_2 e_{2, P_1} = - D_1 (1 + e_{1, P_1}) \]

\[ e_{2, P_1} = \frac{-D_1}{D_2} \left[ 1 + e_{1, P_1} \right] \]

If \( e_{1, P_1} \to b/c \frac{dx_1}{dP_1} > 0 \) (by assumption) then

the interior term, \( 1 + e_{1, P_1} \), is positive. Both

\( D_1 + D_2 \) are positive. Therefore, the negative, \( \frac{-D_1}{D_2} \),
determines the sign of \( e_{2, P_1} \), \( [e_{2, P_1} < 0] \)

If \( \frac{dx_1}{dP_1} > 0 \) (Giffen), then using the homogeneity of degree \( 0 \)
of compensated demand functions, we showed this implied \( \left[ \frac{dx_2}{dP_1} > 0 \right] \). We then used the Constant Aggregation to

determine that if \( \left[ \frac{dx_2}{dP_1} > 0 \right] \), then \( e_{1, P_1} > 0 \) and

therefore \( e_{2, P_1} < 0 \). If \( e_{2, P_1} < 0 \), then \( \left[ \frac{dx_2}{dP_1} < 0 \right] \)

And therefore we showed that \( \frac{dx_2}{dP_1} \) and \( \frac{dx_2}{dP_1} \)

are opposite signs.
**BONUS QUESTION #1 (Shephard's Lemma)**

\[ E(p_x, p_y, \tilde{u}) \quad (expected \text{ } f_r) \]

\[ X^c(p_x, p_y, \tilde{u}) = \frac{2E(p_x, p_y, \tilde{u})}{2\tilde{u}_X} \quad (Shephard's \text{ } Lemma) \]

**Proof:**

\[ E = \min_{\tilde{u}} \mathcal{L} = \min_{\tilde{u}} \mathbf{p}_X (X^c(p_x, p_y, \tilde{u}) + P_y \Gamma^c(p_x, p_y, \tilde{u}) + \lambda(p_x, p_y, \tilde{u}) (\tilde{u} - u_{X}(p_x, p_y, \tilde{u}), u_{Y}(p_x, p_y, \tilde{u})) \]

\[ \frac{dE}{dp_x} = \frac{dX^c}{dp_x} + p_x \frac{\partial X^c}{\partial p_x} + p_y \frac{\partial X^c}{\partial p_y} + \frac{\partial \lambda}{\partial p_x} \frac{\partial u_{X}}{\partial p_x} + \frac{\partial \lambda}{\partial p_y} \frac{\partial u_{Y}}{\partial p_y} \]

Recall for \( E \), the following hold:

1. \( \tilde{u} - u(x, y) = 0 \) \( (FOC_3) \)
2. \( p_x - \lambda \frac{du_x}{dx} = 0 \) \( (FOC_1) \)
3. \( p_y - \lambda \frac{du_y}{dy} = 0 \) \( (FOC_2) \)

\[ \frac{dE}{dp_x} = \frac{dX^c(p_x, p_y, \tilde{u})}{dp_x} + p_x \frac{\partial X^c}{\partial p_x} - \lambda \frac{du_x}{dx} \frac{\partial X^c}{\partial p_x} + p_y \frac{\partial X^c}{\partial p_y} - \lambda \frac{du_y}{dy} \frac{\partial X^c}{\partial p_y} + \lambda \frac{\partial u_{X}}{\partial p_x} + \lambda \frac{\partial u_{Y}}{\partial p_y} \]

\[ FOC_3 = 0 \]

\[ \frac{dE}{dp_x} = X^c(p_x, p_y, \tilde{u}) + \left( p_x - \lambda \frac{du_x}{dx} \right) \frac{\partial X^c}{\partial p_x} + \left( p_y - \lambda \frac{du_y}{dy} \right) \frac{\partial X^c}{\partial p_y} \]

\[ FOC_1 = 0 \quad \quad \quad FOC_2 = 0 \]

\[ \frac{dE}{dp_x} = X^c(p_x, p_y, \tilde{u}) \]
Expenditure is concave in prices means:

\[ \frac{dE}{dx^2} \leq 0. \]

\[ \frac{d^2E}{dx^2} = \left( \frac{d}{dP_x} \right) \left( \frac{dE}{dP_x} \right) = \frac{2 \chi^c(P_x, P_y, u)}{2P_x} \leq 0 \]

T eaks 1st Law.
\[
V(p_x, p_y, I) = \max \left[ U(x(p_x, p_y, I), y(p_x, p_y, I)) + \lambda(p_x, p_y, I)(I - p_x x(p_x, p_y, I) - p_y y(p_x, p_y, I)) \right]
\]

\[
\frac{dV}{dT} = \frac{dx}{dT} \frac{dV}{dx} + \frac{dy}{dT} \frac{dV}{dy} + 1 = \lambda \frac{p_x}{dT} - \lambda \frac{p_y}{dT}
\]

**Recall:**

**FOC1:** \( \frac{dy}{dx} - \lambda p_x = 0 \)

**FOC2:** \( \frac{dx}{dy} - \lambda p_y = 0 \)

**FOC3:** \( I - p_x x - p_y y = 0 \)

\[
\frac{dV}{dT} = \frac{dx}{dT} \frac{dV}{dx} + \frac{dy}{dT} \frac{dV}{dy} + \lambda \frac{I - p_x x - p_y y}{dT} + \lambda(p_x, p_y, I)
\]

**FOC1:** \( dx \)

**FOC2:** \( dy \)

**FOC3:** \( I - p_x x - p_y y \)

\[
\frac{dV}{dT} = \lambda(p_x, p_y, I) \geq 0
\]

Income will increase utility so long as one consumes non-zero amounts of \( x \) by which is to say \( \lambda > 0 \) if the budget constraint is binding. If it is not binding, then \( \frac{dV}{dT} \) may be zero.