Proof (Appendix, BH, ch.10, p. 259-260)

Constant returns imply Diminishing marginal Products

- Recall that CRS implies linear homogeneity:

$$q = f(K,L)$$

Def'n of CRS: $f(\lambda K, \lambda L) = \lambda f(K,L) = \lambda q$

For a function that is homogeneous of degree 1, Euler's equation can be used. To show this, we start with the following:

$$\text{Def'n of H}^1\text{:} \quad f(\alpha K, \alpha L) = \alpha f(K,L) \qquad (1)$$

(i) Diff. wrt α:

$$\frac{\partial f}{\partial K}\frac{\partial K}{\partial \alpha} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial \alpha} = f(K,L)$$

$$\boxed{f_K K + f_L L = f} \qquad (2)$$

(ii) We know diff. wrt L. Rewrite eq. (2) first though:-

$$f_K(K,L)K + f_L(K,L)L = f(K,L)$$

$$f_{KL}K + f_L + f_{LL}L = f_L$$

$$f_{LL}L + f_{KL}K + f_L = f_L \qquad (3)$$

Now, collecting terms in (3),

$$f_{LL}L + f_{KL}K = 0$$

$$\Rightarrow f_{LL}L = -f_{KL}K$$

$$\Rightarrow \boxed{f_{LL} = -f_{KL}\left(\frac{K}{L}\right)} \qquad (4)$$

(iii) Now, diff. eq. (2) wrt K:

$$f_{KK}K + f_K + f_{LK}L = f_K$$

$$\Rightarrow f_{KK}K + f_{LK}L = \cancel{f_K}\ 0$$

$$\Rightarrow f_{KK}K = -f_{LK}L$$

$$\Rightarrow \boxed{f_{KK} = -f_{LK}\left(\frac{L}{K}\right)} \qquad (5)$$

(iv) Next, totally diff. the production fn. along an isoquant:

$$df = 0. \qquad \text{(isoquant)}$$

$$f(K, L) \qquad \text{(production fn)}$$

$$f_K\,dK + f_L\,dL = 0 \qquad \text{(Diff.)}$$

$$f_K\,dK = -f_L\,dL$$

$$\boxed{\frac{dK}{dL} = -\frac{f_L}{f_K}} \qquad (6)$$

Now, diff. eq. (6) wrt L; recall for a single isoquant, $K = K(L)$:

$$\frac{dK}{dL} = -\frac{f_L(K, L)}{f_K(K, L)} = \frac{-f_L(K[L], L)}{f_K(K[L], L)}$$

$$\cancel{\frac{d^2K}{dL^2} = \frac{[f_{LL} f_K - f_{KL} f_L]}{f_K^2}}$$

$$\frac{d^2K}{dL^2} = -\frac{\left[\left(f_{LK}\frac{dK}{dL} + f_{LL}\right)f_K - \left(f_{KK}\frac{dK}{dL} + f_{KL}\right)f_L\right]}{f_K^2}$$

$$= \left(\frac{-1}{f_K^2}\right)\left(f_{LK}\frac{dK}{dL} f_K + f_{LL} f_K - f_{KK}\frac{dK}{dL} f_L - f_{KL} f_L\right)$$

Subst. $\frac{dK}{dL} = \frac{-f_L}{f_K}$ (eq. 6)

$$= \left(\frac{-1}{f_K^2}\right)\left(f_{LK}\cancel{f_K}\frac{-f_L}{\cancel{f_K}} + f_{LL} f_K - f_{KK} f_L\left(\frac{-f_L}{f_K}\right) - f_{KL} f_L\right)$$

$$= \left(\frac{-1}{f_K^2}\right)\left(-f_{LK} f_L + f_{LL} f_K + f_{KK} f_L^2 f_K - f_{KL} f_L\right) \qquad (7)$$

Young's theorem: $\boxed{f_{LK} = f_{KL}}$ (8)

$$\frac{d^2K}{dL^2} = \left(\frac{-1}{f_K^3}\right)\left(\boxed{f_{LL}}\,f_K^2 - 2f_{KL}f_Lf_K + \boxed{f_{KK}}\,f_L^2\right) > 0 \quad (9)$$

Note, diminishing MRTS is based on this, as $\frac{d^2K}{dL^2} > 0$

Recall the linear homogeneous functions derived earlier using equation 2:

$$\text{eq. (4)} \quad \boxed{f_{LL}} = -f_{KL}\left(\frac{K}{L}\right)$$

$$\text{eq. (5)} \quad \boxed{f_{KK}} = -f_{LK}\left(\frac{L}{K}\right)$$

Substitute (4) + (5) into (9)

$$\frac{d^2K}{dL^2} = \left(\frac{-1}{f_K^3}\right)\left(-f_{KL}\left(\frac{K}{L}\right)f_K^2 - 2f_{KL}f_Lf_K + (-f_{LK})\left(\frac{L}{K}\right)f_L^2\right)$$

$$= \frac{f_{KL}\left(\frac{K}{L}\right)f_K^2 + \boxed{2f_{KL}f_Lf_K} + f_{LK}\left(\frac{L}{K}\right)f_L^2}{f_K^3} \quad (10)$$

Let $\boxed{2f_{KL}f_Lf_K = f_{KL}f_Lf_K\left(\frac{K}{K}\right) + f_{LK}f_Kf_L\left(\frac{L}{L}\right)}$

$$\frac{d^2K}{dL^2} = \frac{f_{KL}f_K^2\left(\frac{K}{L}\right) + f_{KL}f_Lf_K\left(\frac{K}{K}\right) + f_{LK}f_Kf_L\left(\frac{L}{L}\right) + f_{LK}f_L^2\left(\frac{L}{K}\right)}{f_K^3} \quad (11)$$

Collect terms on left + right of (11) RHS:

$$\frac{d^2K}{dL^2} = \frac{\frac{f_{KL}f_K}{L}\left(f_KK + f_LL\right) + \frac{f_{LK}f_L}{K}\left(f_KK + f_LL\right)}{f_K^3} \quad (12)$$

Finally, recall that eq (2), $f_KK + f_LL = f$

$$\cancel{\frac{d^2K}{dL^2} = \frac{f\,f_{KL}f_K}{Lf_K^3} + \frac{f\,f_{LK}f_L}{Kf_K^3}}$$

$$\frac{d^2K}{dL^2} = \left[\frac{f_{KL} f_K}{L} f + \frac{f_L f_{KL}}{K}(f_K K + f_L L)\right]\frac{1}{f_K^3}$$

$$= \left[\frac{f f_{KL} f_K K}{LK} + \frac{f_L L f_{KL}(f_K K + f_L L)}{KL}\right]\frac{1}{f_K^3}$$

$$= \frac{f f_{KL} f_K K + (f_K K + f_L L) f_{KL} f_L L}{f_K^3 KL}$$

$$= \frac{f_{KL}(f f_K K + f f_L L)}{f_K^3 KL}$$

$$= \frac{f_{KL}(f [f_K K + f_L L])}{f_K^3 KL} \quad (13)$$

Again, Substitute (2) into (13)

$$\frac{d^2K}{dL^2} = \frac{f_{KL}(f^2)}{f_K^3 KL}$$

$$\frac{d^2K}{dL^2} = \frac{f_{KL} f^2}{f_K^3 KL} \text{, OR}$$

$$\boxed{\frac{d^2K}{dL^2} = \frac{f^2 f_{KL}}{LK f_K^3}} \quad (14)$$

Note, by diminishing MRTS (eg. 9) $\frac{d^2K}{dL^2} > 0$. This means, though, that eg. (14) is also positive. For (14) to be positive, note:

$f^2 > 0$

$L > 0$

$K > 0$

$f_K > 0$; so $f_K^3 > 0$

f_{KL} ? $\longleftarrow$ As $\frac{d^2K}{dL^2} > 0$, $\therefore f_{KL} > 0$.

$$\frac{d^2K}{dL^2} = \frac{f^2 f_{KL}}{LK f_K^3} > 0 \quad (14)$$

If $f_{KL} > 0$, then recall the earlier equations (4) and (5) which also featured f_{KL}:

$$(4) \quad f_{LL} = -f_{KL}\left(\frac{K}{L}\right)$$

$$(5) \quad f_{KK} = -f_{LK}\left(\frac{L}{K}\right)$$

If in fact $f_{LK} > 0$, then both $f_{LL} < 0$ and $f_{KK} < 0$. Therefore, from this proof, diminishing MRTS in production

#

Cost functions in the short-run.

Sometimes a firm cannot expand one or more of the inputs used in production, and as a consequence they face constraints that keep them from achieving the cost minimizing point of production (in the SR).

Recall that when cost has been minimized, the firm chooses $MRTS = w/v$ (for the 2-good case).

Note why this will occur. We redraw the SR expansion path. We will assume here that capital is the fixed input, while labor is variable, but that is just for convenience.

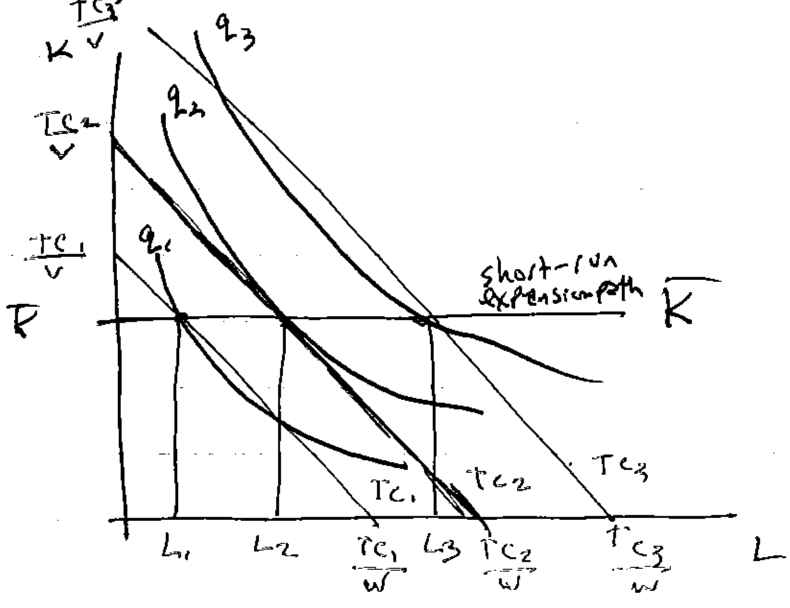
short-run total cost

(1) $$\boxed{SRTC = SRTC(x(L; \bar{K}); \bar{w}, \bar{v})}$$

output, labor, fixed capital, wages, rental rate

Capital is fixed at $\bar{K}$, which implies that if the firm is going to change production, it will only do so using labor, its variable input.

Figure 1



Notice in Figure 1 that the three cost curves — TC_1, TC_2, & TC_3 — do not all lie tangent to their corresponding isoquants. In fact only one of these cost functions is cost minimizing — TC_2 — as it is the only function for which K/L satisfies $MRTS = \frac{w}{v}$. The first total cost fn., TC_1, uses "too much" capital (prove this to yourself). Whereas TC_3 uses "too little" capital. As the total cost function in both q_1 & q_3 case is contained partly within the convex isoquants, we know it is higher overall cost. It is also caused by the firm's inability to freely adjust all inputs.

Definitions

short-run total cost is equal to:

$$\boxed{SRTC = \underbrace{\bar{w} L^c(\bar{K}, \bar{w}, \bar{v}, \bar{q})}_{\text{variable cost}} + \underbrace{\bar{v}\bar{K}}_{\text{fixed cost}}} \quad (1)$$

We can see here that part of SRTC is due to fixed input costs ($\bar{v}\bar{K}$) and part is due to variable input costs ($\bar{w} L^c(\bar{K}, \bar{w}, \bar{v}, \bar{q})$).

Notice, if $\bar{w}L(q=0) = 0$, SRTC is not zero:

Let $q = 0$. (shut-down production).

Rewrite SRTC:

$$SRTC = 0 + \bar{v}\bar{K}$$

$$SRTC = \bar{v}\bar{K}$$

We call $\bar{v}\bar{K}$ "fixed cost" because it is a level of cost which doesn't go away, even under shut-down.

(2) $\boxed{FC = \bar{v}\bar{K}}$

Short-run variable costs are those additional costs spent on production, q, that are due to the use of the variable inputs. ~~If Sets~~ Let's see:

$(q=0) \quad SRTC = \bar{v}\bar{K}$

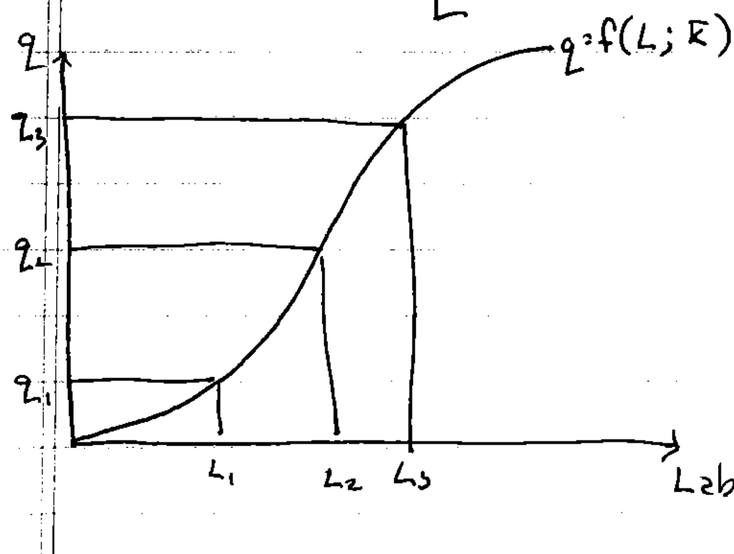
$(q>0) \quad SRTC = \bar{v}\bar{K} + \bar{w}L$

Short-run variable cost therefore is $\bar{w}L$:

(3) $\boxed{SRVC = \bar{w}L}$

To derive the SR variable cost, we start with the firm's "total product of labor" function:

(4) $$q = f(L; \bar{K})$$ "total product of labor"



Depiction of the total product of labor from Figure 1.

Figure 2

Notice that the firm's production ~~[illegible]~~ function yields a total product of labor function that is first convex, then concave.

The inverse of the total product of labor is:

$$(4) \quad \boxed{q_L^{-1} = L = L(q;\bar{K})}$$

And notice, the inverse of total product of labor will be simply the mirror of Figure 2 — first concave, then convex.

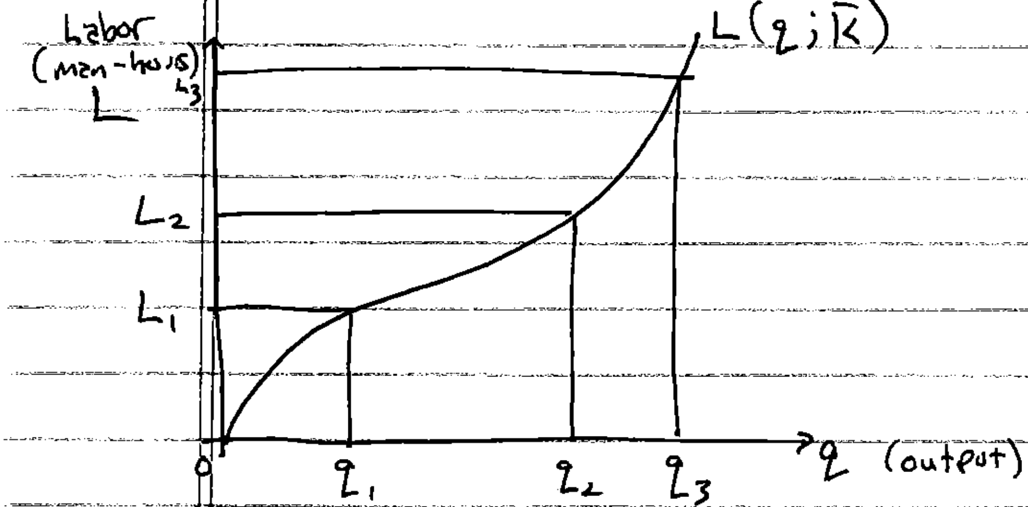


Figure 3 inverse of total product of labor $L = L(q;\bar{K})$

This hopefully makes intuitive why it is that the short-run variable cost curve (SRVC) is proportional to the inverse of the total product of labor, $L(q;\bar{K})$, be recall from eq. (3) on p.3 the definition of SRVC:

$$(3) \quad SRVC = wL$$

$$(3b) \quad \boxed{SRVC = wL(q;\bar{K})}$$

When $q=0$, $SRVC=0$ because at $q=0$, $L(q;\bar{K})=0$, and SRVC is just w times L (which is zero at $q=0$).

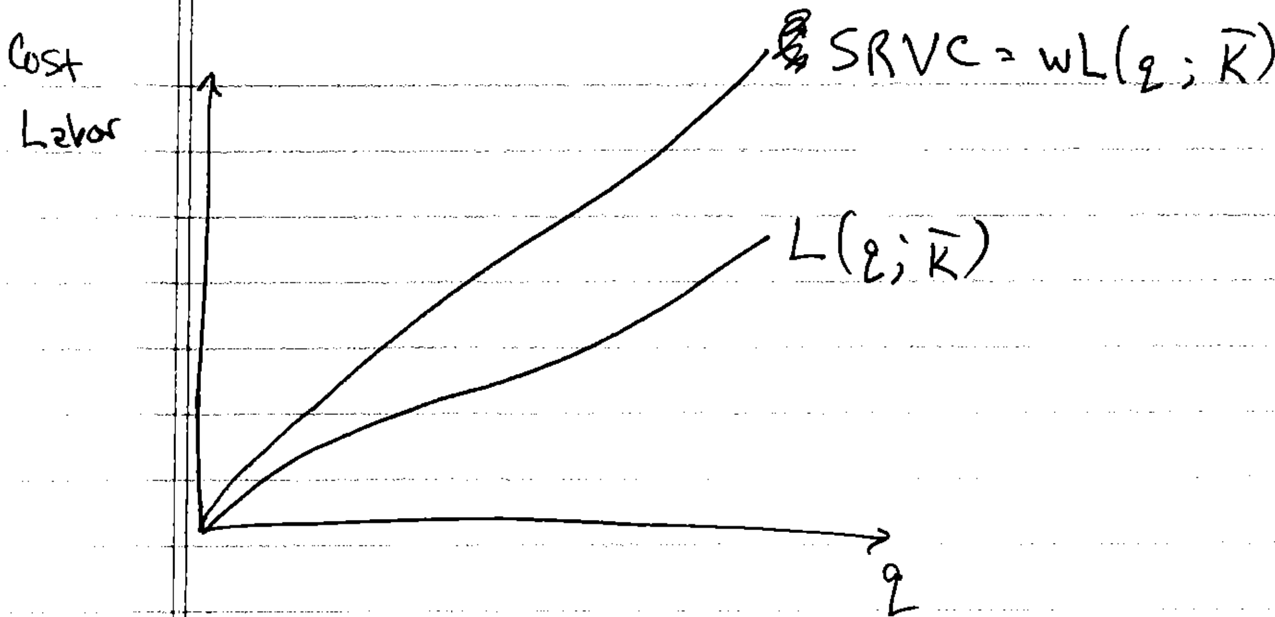
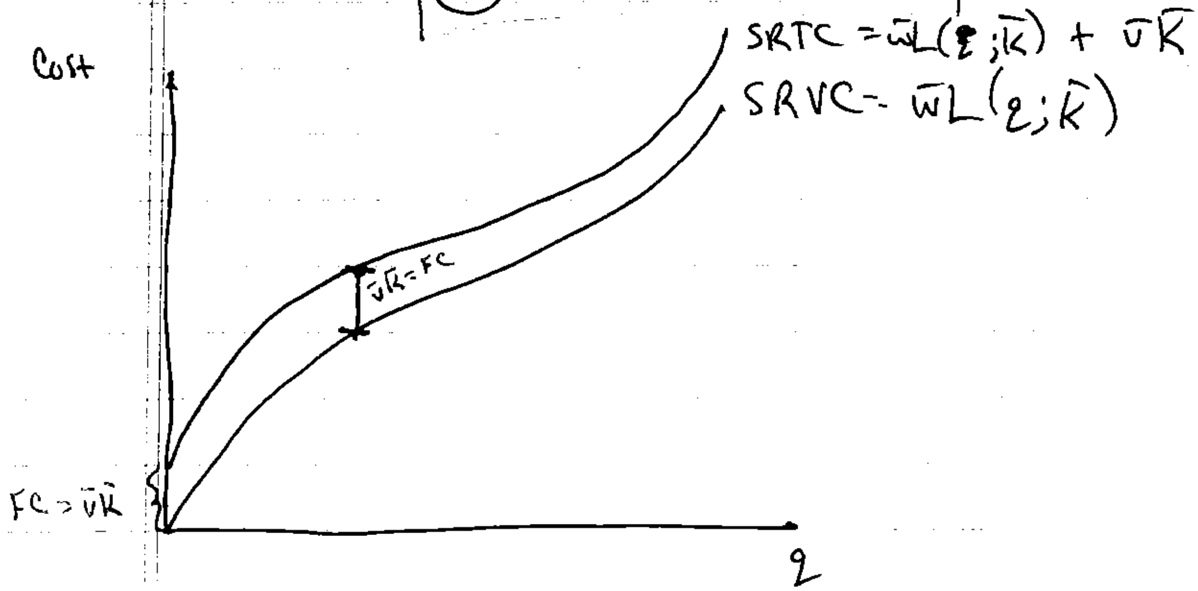


Figure 4: Deriving the Firm's SRVC function

Notice also that the SRTC and SRVC are identical functions differing only by an additive constant, $\bar{v}\bar{K}$:

$$(1)\quad SRTC = \bar{w}L + \bar{v}\bar{K}$$

$$\boxed{(1b)\quad SRTC = SRVC + FC}$$



~~More definitions~~: Average & marginal Costs in Short-run

Short-run average variable cost (SR AVC):

(5) $$\boxed{SR\,AVC = \frac{\bar{w}L(q)}{q} = \frac{SRVC}{q}}$$

~~Short~~ Average fixed cost (AFC):

(6) $$\boxed{AFC = \frac{\bar{v}\bar{K}}{q} = \frac{FC}{q}}$$

Short-run average total cost (SR ATC):

(7) $$\boxed{SRATC = \frac{SRTC}{q} = \frac{\bar{w}L + \bar{v}\bar{K}}{q} = \boxed{\frac{\bar{w}L}{q} + \frac{\bar{v}\bar{K}}{q}} = SRAVC + AFC}$$

short-run marginal ~~variable~~ cost ~~(SRMC)~~

(8) $$\frac{d}{dq} SRTC = \frac{d}{dq}\left(\bar{w}L(q) + \bar{v}\bar{K}\right) = \frac{d}{dq}(\bar{w}L(q)) + \frac{d}{dq}(\bar{v}\bar{K})$$

$$= \frac{d}{dq}(SRVC) + \frac{d}{dq}(FC)$$

(8) $$\boxed{SR\,MC = \frac{d}{dq}(SRVC)} + 0$$

short-run marginal variable cost (SRMVC):

(9) $$\frac{d}{dq} SRVC = \frac{d}{dq}\left(\bar{w}L(q)\right)$$

examples of each using $q = f(K, L) = (KL)^{1/2}$.

Assume capital is fixed at $\bar{K}$. Rewrite production fn:

$$q = \bar{K}^{1/2} L^{1/2} \qquad \text{"total product of labor"}$$

We can therefore find the inverse of the total product of labor as:

$$q^{-1} = L(q; \bar{K})$$
$$\Rightarrow L^{1/2} = \frac{q}{\bar{K}^{1/2}}$$
$$\Rightarrow L = \frac{q^2}{\bar{K}}$$
$$\Rightarrow \boxed{L(q; \bar{K}) = \frac{q^2}{\bar{K}}} \qquad \text{"inverse of total product of labor"}$$

Now use this to find all of the cost functions we just discussed:

$$\rightarrow SRAVC = \frac{\bar{w} L(q)}{q} = \frac{\bar{w}}{q} \cdot \frac{q^2}{\bar{K}} = \frac{\bar{w} q}{\bar{K}}$$

$$\boxed{SRAVC = \frac{\bar{w} q}{\bar{K}}}$$

$SRVC = \bar{w} L(q)$

$\boxed{SAVC = \frac{\bar{w} q^2}{\bar{K}}}$

$$\rightarrow \boxed{AFC = \frac{\bar{v} \bar{K}}{q}}$$

$$\Rightarrow SRTC = wL + v\bar{K}$$
$$= \bar{v}\bar{K} + w\left[\frac{q^2}{\bar{K}}\right] = \bar{v}\bar{K} + \frac{\bar{w} q^2}{\bar{K}}$$

$$\Rightarrow SRTC = \bar{v}\bar{K} + \bar{w}\, q^2/\bar{K}$$

$$SRATC = \frac{SRTC}{q} = \frac{\bar{w}L + \bar{v}\bar{K}}{q} = \frac{\bar{w}L(q)}{q} + \frac{\bar{v}\bar{K}}{q}$$

$$\boxed{SRATC = \frac{\bar{w}\,q}{\bar{K}} + \frac{v\bar{K}}{q}}$$

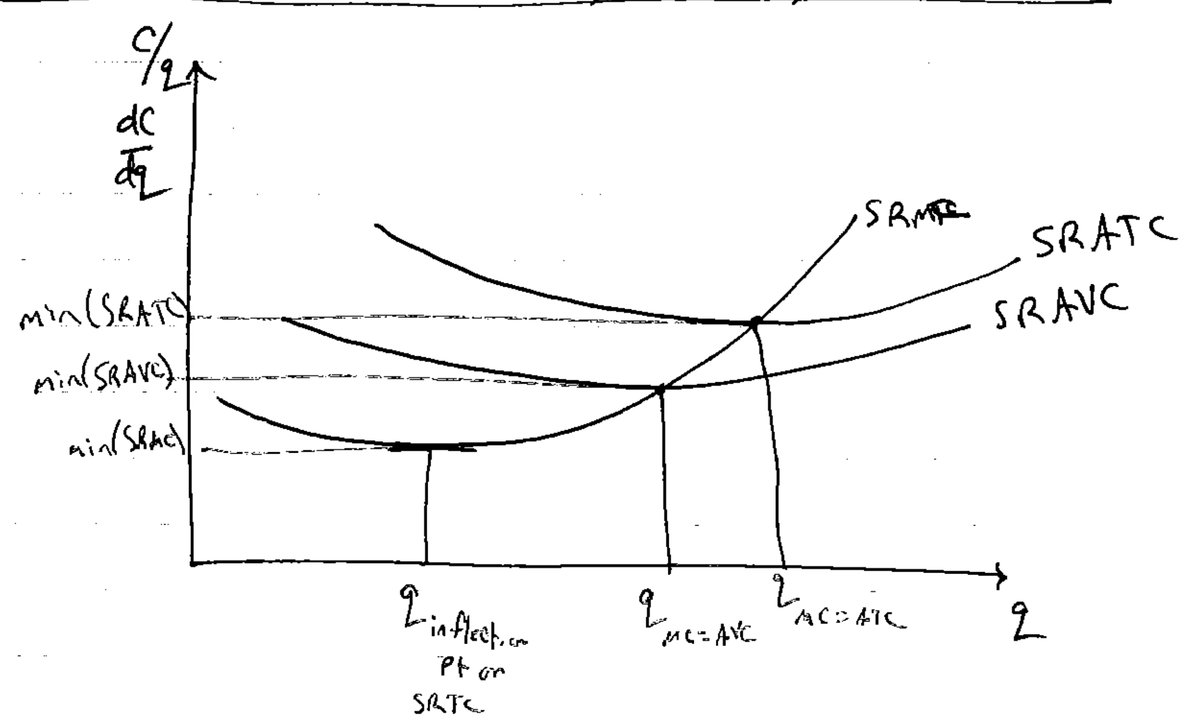
$$SRMC = \frac{d}{dq} SRTC = \frac{d}{dq} SRVC$$

$$= \frac{d}{dq}\left(wL(q)\right)$$

$$= \frac{d}{dq}\left[\frac{\bar{w}\,q^2}{\bar{K}}\right]$$

$$\boxed{SRMC = \frac{2\bar{w}\,q}{\bar{K}}}$$

Relationships b/w SRATC, SRAVC, and SRMC:



Note that the relationship described here for a production function w/ H' in K&L that has fixed capital in SR will ~~[illegible]~~ exhibit ~~[illegible]~~ rising MC. To show this analytically, we minimize SRATC and show that at the output level, q, that minimizes average cost, the two cost functions are equivalent:

$$SRATC = \frac{SRTC}{q} = \frac{SR(q; \bar{w}, \bar{v}; \bar{K})}{q}$$

$$SRATC = \frac{\bar{v}\bar{K}}{q} + \frac{\bar{w}q}{\bar{K}}$$

$$\frac{dSRATC}{dq} = -\frac{\bar{v}\bar{K}}{q^2} + \frac{\bar{w}}{\bar{K}} = 0$$

$$\cancel{\frac{\bar{K}}{\bar{w}}}\, q^2\, \cancel{\frac{\bar{w}}{\bar{K}}} = \frac{\bar{v}\bar{K}}{\cancel{q^2}}\, \cancel{q^2}\, \frac{\bar{K}}{\bar{w}}$$

$$q^2 = \frac{\bar{v}\bar{K}^2}{\bar{w}}$$

SRATC is minimized at $q_{\substack{min\\ ATC}}$ $\left| q_{\substack{min\\ ATC}} = \left(\frac{\bar{v}}{\bar{w}}\right)^{1/2}\bar{K} \right)$

~~Now, calculate SRMC:~~

Now, plug $q_{\substack{min\\ ATC}}$ into the SRATC function:

$$SRATC = \frac{\bar{v}\bar{K}}{q_{\substack{min\\ ATC}}} + \frac{\bar{w}}{\bar{K}}\left(q_{min\text{-}ATC}\right)$$

$$= \frac{\bar{v}\cancel{\bar{K}}}{\left(\frac{\bar{v}}{\bar{w}}\right)^{1/2}\cancel{\bar{K}}} + \frac{\bar{w}}{\bar{K}}\left(\frac{\bar{v}}{\bar{w}}\right)^{\frac{1}{2}}\bar{K}$$

$SRATC_{min}$: Substitute $q_{min\,ATC}$:

$$SRATC_{min} = \frac{\bar{V}\bar{K}}{q_{min\,ATC}} + \frac{\bar{W}L}{q_{min\,ATC}} = \frac{\bar{V}\cancel{\bar{K}}}{\cancel{\bar{K}}\left(\frac{\bar{V}}{\bar{W}}\right)^{1/2}} + \frac{\bar{W}}{\cancel{\bar{K}}}\left(\frac{\bar{V}}{\bar{W}}\right)^{\frac{1}{2}}\cancel{\bar{K}}$$

$$= \frac{\bar{V}\bar{W}^{1/2}}{\bar{V}^{1/2}} + \frac{W\bar{V}^{1/2}}{\bar{W}^{\frac{1}{2}}}$$

$$= (\bar{V}\bar{W})^{\frac{1}{2}} + (\bar{V}\bar{W})^{\frac{1}{2}}$$

$$\boxed{SRATC_{min} = 2(\bar{V}\bar{W})^{\frac{1}{2}}}$$

Now insert $q_{min\,ATC}$ into the SRMC function:

$$SRMC = \frac{2\bar{W}q}{\bar{K}} = \frac{2\bar{W}}{\cancel{\bar{K}}}\left(\frac{\bar{V}}{\bar{W}}\right)^{\frac{1}{2}}\cancel{\bar{K}}$$

$$= \frac{2\bar{W}\bar{V}^{\frac{1}{2}}}{\bar{W}^{\frac{1}{2}}}$$

$$SRMC = 2(\bar{W}\bar{V})^{\frac{1}{2}}$$

$$\boxed{SRMC = 2(\bar{V}\bar{W})^{\frac{1}{2}}}$$

<u>Note</u>: when SRATC is minimized, SRMC = SRATC. Same for the other ~~@~~ min & SRAVC.

<u>Diminishing Returns</u>

Both functions which are described by constant returns to scale and decreasing returns to scale have diminishing marginal products: $f_{LL} < 0$;
$f_{KK} < 0$

Implications Diminishing Returns (cont.)

with one variable input (eg, Labor), three important properties of the SR cost functions follow from assuming diminishing returns to variable inputs. And since both constant + ~~diminishing~~ decreasing returns to scale have diminishing ~~MPs~~ marginal products, this is also the property of the constant + decreasing return functions as well.

Property #1 $\boxed{\text{1. SR MC is increasing with output.}}$

Recall the definition of SRMC:

$$SRMC = w\frac{dL(q)}{dq}$$

For this, let $L(q) = q^{-1}$. Therefore, change it to:

$$SRMC = \frac{w}{\frac{dq(L)}{dL}} = \frac{w}{MP_L}$$

$$\boxed{SRMC = \frac{w}{MP_L}}$$

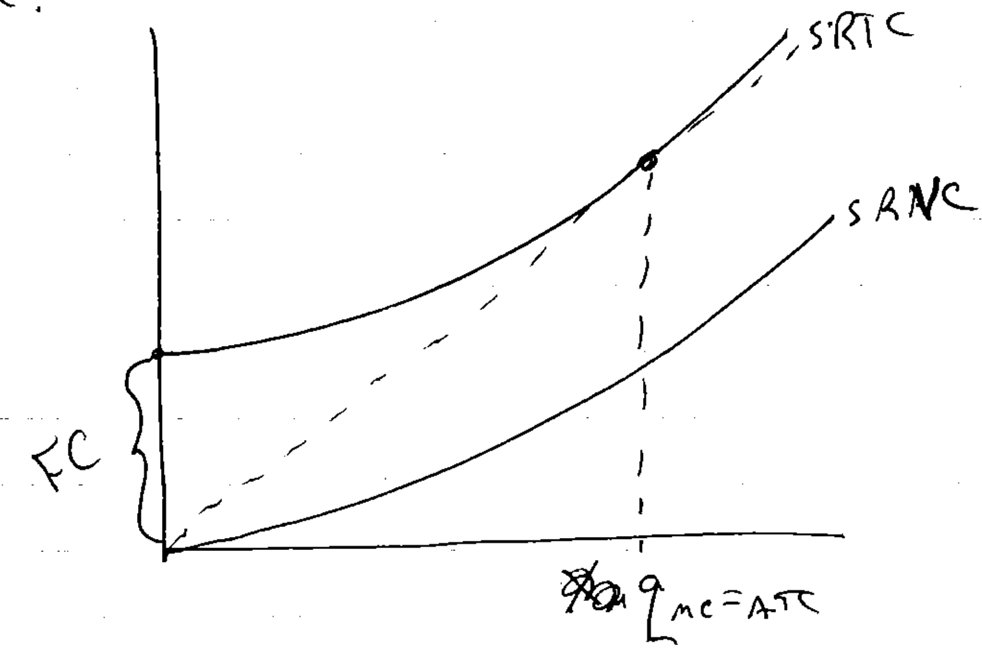
If the ~~SR~~ MP_L is declining ($\frac{d^2q(L)}{dL^2} < 0 = f_{LL} < 0$), then we can see SRMC must be rising since the firm is employing more labor at wage, $\overline{w}$, to produce each additional output, q.

Property #3: The SRATC function is "U-shaped".

Property #1 says that SRMC is rising, which implies the SRVC is convex. Since the ~~SRATC~~ SRTC is above the SRVC by an amount equalling FC, this implies a U-shapeble:

$$SRATC = AFC + SRAVC$$

where at low levels of output, AFC is the larger portion of SRATC, but at higher points, SRAVC is. And since SRVC is convex, SRAVC will eventually rise thus pulling up SRATC as well: The SRAVC may be convex from the origin with no minimum AVC. But, when FC is added to the VC function, the SRTC function will have an intercept at an amount equal to FC, which implies that there is a line from the origin tangent to SRTC and a minimum SRATC.



Short-run costs with more than one variable input.

Say the firm is using variable materials (m) at price, p_m, as well as labor & capital, but $K = \bar{K}$.

$\bar{K}$: capital fixed
m : quantity of materials
p_m : price of material inputs

Then the $SRVC = p_m m + wL$, and
$$q = f(L, m; \bar{K}, \bar{w}, \bar{v}, \bar{p}_m)$$

Cost-min problem here is:

$$\min \ \bar{v}\bar{K} + \bar{w}L + \bar{p}_m m$$
$$\text{s.t. } q = f(\bar{K}, L, m)$$

$$\min \ \mathcal{L} = \bar{v}\bar{K} + \bar{w}L + \bar{p}_m m + \lambda(\bar{q} - f(\bar{K}, L, m))$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda f_L = 0 \ ; \ \lambda = \frac{w}{f_L}$$

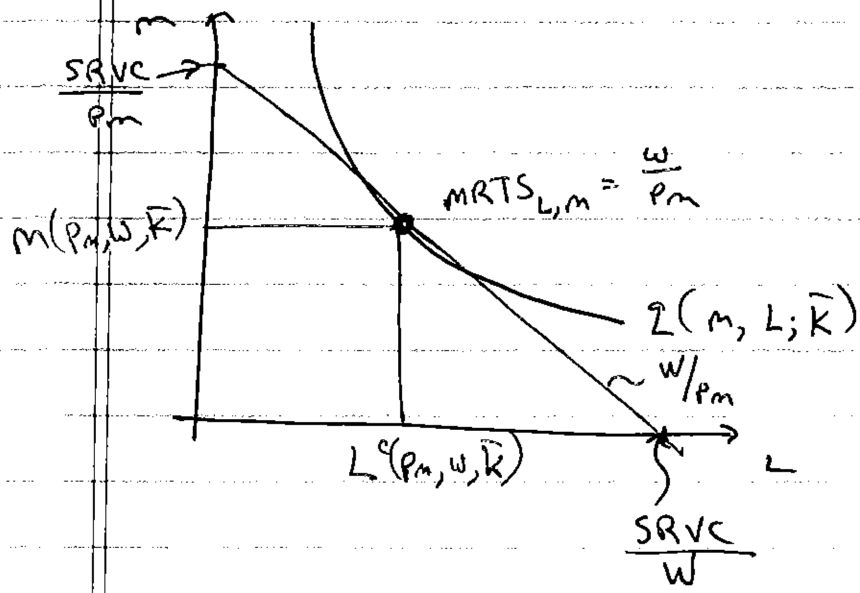
$$\frac{\partial \mathcal{L}}{\partial m} = p_m - \lambda f_m = 0 \ ; \ \lambda = \frac{p_m}{f_m}$$

$$\frac{w}{f_L} = \frac{p_m}{f_m}$$

$$\frac{w}{p_m} = \frac{f_L}{f_m} = \frac{MP_L}{MP_m}$$

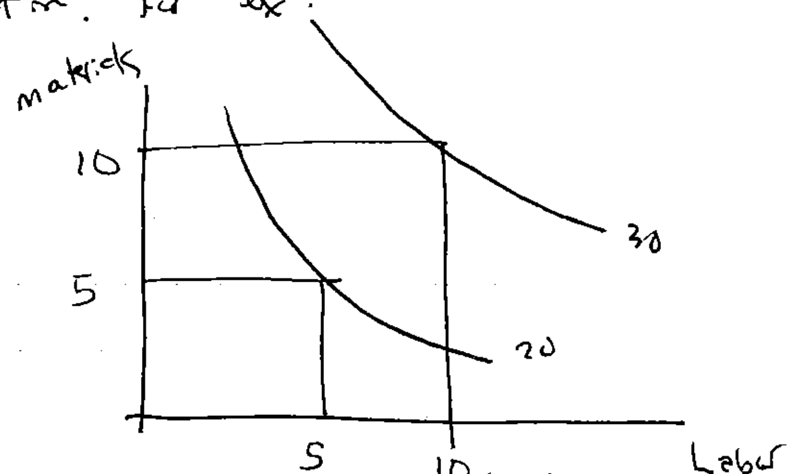
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - f(\bar{K}, L, m) = 0$$

Now, this is equivalent to choosing combo of Labor & materials that will minimize SRVC.



If more than one variable input is used in production, diminishing MP of the individual inputs no longer guarantees increasing SRMC functions or U-shaped ATC functions.

Rather, production functions must exhibit diminishing returns to all variable inputs in combination. For ex:



Doubling of variable inputs (100%) yielded 50% change in output. Since variable inputs are rising faster than output, MC is rising. And once MC is rising, the SRATC is U-shaped.