

Second, option theory gives a simple, powerful framework for describing complex decision trees. For example, suppose that you have the option to abandon an investment. The complete decision tree would overflow the largest classroom chalkboard. But now that you know about options, the opportunity to abandon might be summarized as “an American put.” Of course, not all real problems have such easy option analogies, but we can often approximate complex decision trees by some simple package of assets and options. A custom decision tree may get closer to reality, but the time and expense may not be worth it. Most men buy their suits off the rack even though a custom-made Armani suit would fit better and look nicer.

22.3 THE BLACK-SCHOLES FORMULA

Look back at Figure 22.1, which showed what happens to the distribution of possible Genentech stock price changes as we divide the option’s life into a larger and larger number of increasingly small subperiods. You can see that the distribution of price changes becomes increasingly smooth.

If we continued to chop up the option’s life in this way, we would eventually reach the situation shown in Figure 22.4, where there is a continuum of possible stock price changes at maturity. Figure 22.4 is an example of a lognormal distribution. The lognormal distribution is often used to summarize the probability of different stock price changes.⁹ It has a number of good commonsense features. For example, it recognizes the fact that the stock price can never fall by more than 100%, but that there is some, perhaps small, chance that it could rise by much more than 100%.

Subdividing the option life into indefinitely small slices does not affect the principle of option valuation. We could still replicate the call option by a levered



FIGURE 22.4

As the option’s life is divided into more and more subperiods, the distribution of possible stock price changes approaches a lognormal distribution.

⁹When we first looked at the distribution of stock price changes in Chapter 9, we depicted these changes as normally distributed. We pointed out at the time that this is an acceptable approximation for very short intervals, but the distribution of changes over longer intervals is better approximated by the lognormal.

investment in the stock, but we would need to adjust the degree of leverage continuously as time went by. Calculating option value when there is an infinite number of subperiods may sound a hopeless task. Fortunately, Black and Scholes derived a formula that does the trick.¹⁰ It is an unpleasant-looking formula, but on closer acquaintance you will find it exceptionally elegant and useful. The formula is

$$\text{Value of call option} = [\text{delta} \times \text{share price}] - [\text{bank loan}]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ [N(d_1) \times P] & - & [N(d_2) \times \text{PV}(\text{EX})] \end{array}$$

where

$$d_1 = \frac{\log[P/\text{PV}(\text{EX})]}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$N(d)$ = cumulative normal probability density function¹¹

EX = exercise price of option; PV(EX) is calculated by discounting at the risk-free interest rate r_f

t = number of periods to exercise date

P = price of stock now

σ = standard deviation per period of (continuously compounded) rate of return on stock

Notice that the value of the call in the Black-Scholes formula has the same properties that we identified earlier. It increases with the level of the stock price P and decreases with the present value of the exercise price PV(EX), which in turn depends on the interest rate and time to maturity. It also increases with the time to maturity and the stock's variability ($\sigma\sqrt{t}$).

To derive their formula Black and Scholes assumed that there is a continuum of stock prices, and therefore to replicate an option investors must continuously adjust their holding in the stock. Of course this is not literally possible, but even so the formula performs remarkably well in the real world, where stocks trade only intermittently and prices jump from one level to another. The Black-Scholes model has also proved very flexible; it can be adapted to value options on a variety of assets such as foreign currencies, bonds, and commodities. It is not surprising, therefore, that it has been extremely influential and has become the standard model for valuing options. Every day dealers on the options exchanges use this formula to make huge trades. These dealers are not for the most part trained in the formula's mathematical derivation; they just use a computer or a specially programmed calculator to find the value of the option.

Using the Black-Scholes Formula

The Black-Scholes formula may look difficult, but it is very straightforward to apply. Let us practice using it to value the Genentech call.

¹⁰ The important assumptions of the Black-Scholes formula are that (a) the price of the underlying asset follows a lognormal random walk, (b) investors can adjust their hedge continuously and costlessly, (c) the risk-free rate is known, and (d) the underlying asset does not pay dividends.

¹¹ That is, $N(d)$ is the probability that a normally distributed random variable \bar{x} will be less than or equal to d . $N(d_1)$ in the Black-Scholes formula is the option delta. Thus the formula tells us that the value of a call is equal to an investment of $N(d_1)$ in the common stock less borrowing of $N(d_2) \times \text{PV}(\text{EX})$.

Here are the data that you need:

- Price of stock now = $P = 80$
- Exercise price = $EX = 80$
- Standard deviation of continuously compounded annual returns = $\sigma = .4068$
- Years to maturity = $t = .5$
- Interest rate per annum = $r_f = 5\%$ (or about 2.5% for six months).¹²

Remember that the Black–Scholes formula for the value of a call is

$$[N(d_1) \times P] - [N(d_2) \times PV(EX)]$$

where

$$d_1 = \log[P/PV(EX)]/\sigma\sqrt{t} + \sigma\sqrt{t}/2$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$N(d)$ = cumulative normal probability function

There are three steps to using the formula to value the Genentech call:

Step 1 Calculate d_1 and d_2 . This is just a matter of plugging numbers into the formula (noting that “log” means *natural* log):

$$\begin{aligned} d_1 &= \log[P/PV(EX)]/\sigma\sqrt{t} + \sigma\sqrt{t}/2 \\ &= \log[80/(80/1.025)]/(.4068 \times \sqrt{.5}) + .4068 \times \sqrt{.5}/2 \\ &= .2297 \end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{t} = .2297 - .4068 \times \sqrt{.5} = -.0580$$

Step 2 Find $N(d_1)$ and $N(d_2)$. $N(d_1)$ is the probability that a normally distributed variable will be less than d_1 standard deviations above the mean. If d_1 is large, $N(d_1)$ is close to 1.0 (i.e., you can be almost certain that the variable will be less than d_1 standard deviations above the mean). If d_1 is zero, $N(d_1)$ is .5 (i.e., there is a 50% chance that a normally distributed variable will be below the average).

The simplest way to find $N(d_1)$ is to use the Excel function NORMSDIST. For example, if you enter NORMSDIST(.2297) into an Excel spreadsheet, you will see that there is a .5908 probability that a normally distributed variable will be less than .2297 standard deviations above the mean. Alternatively, you can use a set of normal probability tables such as those in Appendix Table 6. You can see that if $d_1 = .23$, then $N(d_1) = .5910$, quite close to the value that you need.

Again you can use the Excel function to find $N(d_2)$. If you enter NORMSDIST(-.0580) into an Excel spreadsheet, you should get the answer .4769. In other words, there is a probability of .4769 that a normally distributed variable will be less than .0580 standard deviations *below* the mean. Alternatively, if you

¹² In our binomial example, we assumed an interest rate of 2.5% for six months, equivalent to $1.025^2 - 1 = .05063$, or 5.063% annually compounded. Thus $PV(EX) = 80/1.05063^2 = 80/1.025 = \78.05 .

When valuing options, it is more common to use continuously compounded rates (see Section 3.4). If the annual rate is 5.063%, the equivalent continuously compounded rate is 4.939%. (The natural log of 1.05063 is .04939 and $e^{.04939} = 1.05063$.) Using continuous compounding, $PV(EX) = 80 \times e^{-.2 \times .04939} = \78.05 . Both methods give the same answer.

There is only one trick here. If you are using a spreadsheet or computer program that calls for a continuously compounded rate, make sure that this is what you enter. The error if you use the wrong rate will usually be small, but you can waste a lot of time trying to trace it.